Set forcing over models of Zermelo or Mac Lane

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Abstract. Over certain transitive models of Z, the usual treatment of forcing goes awry. But the provident closure of any such set is a provident model of Z, over which, as shown in *Provident sets and rudimentary set forcing*, forcing works well. In *The Strength of Mac Lane Set Theory* a process is described of passing from a transitive model of Z + TCo to what is here called its lune, which is a larger model of Z + KP. *Theorem:* Over a provident model of Z, the two operations of forming lunes and generic extensions commute. Corresponding results hold for transitive models of Mac Lane set theory + TCo.

Contents

0: Preliminaries

The axioms of the weak systems we shall consider are, with one exception, summarised in §2 and discussed in detail in *The Strength of Mac Lane Set Theory* [M1]; the single exception being the system `PROV` introduced in [M3].

Rudimentary recursion

Let p be a set. We call a unary function $F : V \longrightarrow V$ *p-rudimentarily recursive* if there is a rudimentary binary function $G : V \longrightarrow V$ such that for all x $F(x) = G(p, F \restriction x)$.

0·0 Example The rank function, $\varrho : V \longrightarrow On$, is $\varnothing$-rudimentarily recursive:

$$\varrho(x) = \bigcup\{\varrho(y) \dotplus 1 \mid y \in x\}.$$

0·1 Example The transitive closure, $\mathrm{tcl}(x)$ of a set is ditto:

$$\mathrm{tcl}(x) = x \cup \bigcup\{\mathrm{tcl}(y) \mid y \in x\}.$$

0·2 Example Ordinal addition is defined thus:

$$\begin{array}{ll} & \alpha + 0 = \alpha \\ & \alpha + (\beta \dotplus 1) = (\alpha + \beta) \dotplus 1 \\ \text{for limit } \lambda \qquad & \alpha + \lambda = \bigcup\{\alpha + \nu \mid \nu < \lambda\} \end{array}$$

For each ordinal α the map $\beta \mapsto \alpha + \beta$ is α-rud rec, but no p exists for which either $\beta \mapsto \beta + \beta$ or $\alpha \mapsto \alpha + \beta$ would be p-rud rec.

0·3 Remark The central example of a rudimentary recursion is the following: let c be transitive. The sequences c_ν and P^c_ν are defined by a rudimentary recursion which should be viewed as simultaneous.

$$\begin{array}{lll} c_0 = \varnothing; \qquad & c_{\nu+1} = c \cap \{x \mid x \subseteq c_\nu\}; \qquad & c_\lambda = \bigcup_{\nu<\lambda} c_\nu; \\ P^c_0 = \varnothing; \qquad & P^c_{\nu+1} = \mathbb{T}(P^c_\nu) \cup c_{\nu+1} \cup \{c_\nu\}; \qquad & P^c_\lambda = \bigcup_{\nu<\lambda} P^c_\nu. \end{array}$$

Here $\mathbb{T}$ is the rudimentary function introduced in [M2]: its chief properties are that for transitive u, $u \subseteq \mathbb{T}(u) \subseteq \mathcal{P}(u)$, $u \in \mathbb{T}(u)$, $\varrho(\mathbb{T}(u)) = \varrho(u) + 1$, and finally that $\bigcup_{n\in\omega}\mathbb{T}^n(u)$ is rud closed, and indeed is the rud closure of $u \cup \{u\}$.

0·4 Remark For any transitive c, $P^c_\omega = \mathbf{HF}$.

Provident sets

We call a set A *p-provident* if it is transitive, non-empty, closed under unordered pairs, and under all p-rud rec functions; so $p \in A$. We call a set A *provident* if it is p-provident for every $p \in A$.

0·5 Example For each ν, the Jensen set J_{ω^ν} is provident. For each $\nu \geqslant 1$, the Jensen set J_ν is $\varnothing$-provident.

0·6 Definition We call an ordinal *indecomposable* if it is infinite and closed under ordinal addition.

0·7 Proposition *For each transitive set c and indecomposable θ, P^c_θ is provident.*

The theory `PROV` is a finitely axiomatisable first order set theory of which the transitive models are precisely the provident sets. These notions are studied in detail in [M3].

0·8 Remark Ordinal multiplication is defined by recursion using ordinal addition: ω, but not ω^2 is a member of J_ω, which implies that ordinal multiplication is not rud rec.

Rudimentary forcing

A development of forcing suitable for models of weak systems of set theory is given in *Provident sets and rudimentary set forcing* [M4]. That approach to forcing derives from but is not identical to that of Shoenfield and Kunen.

The first step to defining the forcing relation is to define

$$p \Vdash_0 \underline{a} \mathrel{\epsilon} \underline{b} \iff_{\rm df} (p,\, a) \in b.$$

$$p \Vdash_1 \underline{a} \mathrel{\epsilon} \underline{b} \iff_{\rm df} \exists q{:}{\in}\mathbb{P}\ \big(p \leqslant q \ \&\ q \Vdash_0 \underline{a} \mathrel{\epsilon} \underline{b}\big).$$

One then defines $p \Vdash \underline{a} = \underline{b}$ by a recursion which may be viewed as essentially rud rec, and is discussed in detail in [M4]; from that one can define $p \Vdash \underline{a} \mathrel{\epsilon} \underline{b}$, and then proceed as usual to define $p \Vdash \varphi$ for all $\dot{\Delta}_0$ sentences φ of the language of forcing.

Now given a transitive set M with $\mathbb{P} \in M$ and an $(M, \mathbb{P})$ generic filter $\mathcal{G}$, we define the associated valuation $\mathrm{val}_{\mathcal{G}}(\cdot)$ by a $\mathcal{G}$-rudimentary recursion:

$$\mathrm{val}_{\mathcal{G}}(x) = \{\mathrm{val}_{\mathcal{G}}(y) \mid \exists p{:}{\in}\mathcal{G}\ (p,\, y) \in x\}.$$

The generic extension $M[\mathcal{G}]$ is then defined to be $\{\mathrm{val}_{\mathcal{G}}(x) \mid x \in M\}$.

In [M4] the following is proved:

0·9 THEOREM *A set-generic extension of a provident set is provident.*

Some remarks about names

0·10 DEFINITION $\mathcal{A}_1(a) \ =_{\rm df}\ \{(p, x) \mid p \Vdash_1 \underline{x} \mathrel{\epsilon} \underline{a}\}$.

As in [M4, section 5], we may check that $\mathcal{A}_1(\mathcal{A}_1(a)) = \mathcal{A}_1(a)$ and that $\mathbf{1} \Vdash \underline{\mathcal{A}_1(a)} = \underline{a}$, so that $\mathcal{A}_1(a)$ and a name the same object in the generic extension; but often $\mathcal{A}_1(a)$ is the more convenient name. In particular, as we now show, the subsets of $\mathcal{A}_1(a)$ in the ground model form names for the subsets of $\mathrm{val}_{\mathcal{G}}(a)$ in the generic extension.

0·11 LEMMA *If $X \subseteq \mathcal{A}_1(a)$ and $X \in V$ then $\Vdash \underline{X} \mathrel{\dot\subseteq} \underline{a}$.*

0·12 PROPOSITION *Let $B = \{(s,\, \alpha) \in \mathcal{A}_1(a) \mid s \Vdash \varphi[\underline{\alpha}, \underline{c}]\}$. If $B \in V$, then $\mathbf{1} \Vdash \underline{B} = \underline{a} \mathbin{\dot\cap} \dot{\{}\mathfrak{x} \dot{|} \varphi(\mathfrak{x}; \underline{c}] \dot{\}}$*

Proof: Evidently $s \Vdash_0 \underline{\alpha} \mathrel{\epsilon} \underline{B} \implies s \Vdash \varphi(\underline{\alpha}, \underline{c})$, whence $s \Vdash_1 \underline{\alpha} \mathrel{\epsilon} \underline{B} \implies s \Vdash \varphi(\underline{\alpha}, \underline{c})$. Now suppose $s \Vdash \underline{\alpha} \mathrel{\epsilon} \underline{B}$. Then

$$\forall t{:}{\leq}s\ \exists r{:}{\leq}t\ \exists \beta\ r \Vdash \underline{\beta} = \underline{\alpha} \ \&\ r \Vdash_1 \underline{\beta} \mathrel{\epsilon} \underline{B}.$$

For such r and β $r \Vdash \varphi(\underline{\beta}, \underline{c})$, and so $r \Vdash \varphi(\underline{\alpha}, \underline{c})$. As such r are dense below s, $s \Vdash \varphi(\underline{\alpha}, \underline{c})$.

In the other direction, suppose $p \Vdash \underline{x} \mathrel{\epsilon} \underline{a} \ \&\ \varphi(\underline{x}, \underline{c})$. Then

$$\forall q{:}{\leq}p\ \exists r{:}{\leq}q\ \exists \alpha\, r \Vdash \underline{\alpha} = \underline{x} \ \&\ r \Vdash_1 \underline{\alpha} \mathrel{\epsilon} \underline{a}.$$

For such r and α, $(r,\, \alpha) \in \mathcal{A}_1(a)$, and $r \Vdash \varphi(\underline{\alpha}, \underline{c})$, so $(r,\, \alpha) \in B$, $r \Vdash \underline{\alpha} \mathrel{\epsilon} \underline{B}$ and $r \Vdash \underline{x} \mathrel{\epsilon} \underline{B}$. ⊣ (0·12)

0·13 DEFINITION For any a, Y, set $a \cap^{\mathbb{P}} Y \ =_{\rm df}\ \{(p,\, x) \in \mathcal{A}_1(a) \mid p \Vdash \underline{x} \mathrel{\epsilon} \underline{Y}\}$.

0·14 COROLLARY $\Vdash \underline{a \cap^{\mathbb{P}} Y} = \underline{a} \mathbin{\dot\cap} \underline{Y}$.

For the rest of this section let A be such that $A = \mathcal{A}_1(A)$, as it will be if of the form $\mathcal{A}_1(a)$.

0·15 DEFINITION We call a subset B of A *neat in A* if $[(p,\, x) \in A \ \&\ p \Vdash \underline{x} \mathrel{\epsilon} \underline{B}] \implies (p,\, x) \in B$; equivalently, if $B = A \cap^{\mathbb{P}} B$;

0·16 REMARK If B is neat in A, then

(i) $p \leqslant q \ \&\ (q,\, x) \in B \implies (p,\, x) \in B$;

(ii) $[(p,\, x) \in B \ \&\ (p,\, y) \in A \ \&\ p \Vdash \underline{x} = \underline{y}] \implies (p,\, y) \in B$;

0·17 REMARK If B is neat in A, then for $(p,\, x) \in A$, $p \Vdash_0 \underline{x} \mathrel{\epsilon} \underline{B} \iff p \Vdash_1 \underline{x} \mathrel{\epsilon} \underline{B} \iff p \Vdash \underline{x} \mathrel{\epsilon} \underline{B}$.

0·18 LEMMA *If B and C are disjoint sets both neat in A, then $\Vdash \underline{B} \mathbin{\dot\cap} \underline{C} = \dot{\varnothing}$.*

0·19 REMARK Subsets of A of the form $\{(s,\, \alpha) \in A \mid s \Vdash \varphi[\underline{\alpha}, \underline{C}]\}$ are neat in A.

Now consider the result of forcing over M with the partial order with a unique top point, the ordinal 1, and nothing else, and let $\mathcal{G}_t$ be the generic filter, so that $\mathcal{G}_t = \{1\}$. One would expect to have $M[\mathcal{G}_t] = M$. The usual way to show that $M \subseteq M[\mathcal{G}]$ is to define

$$\hat{x} = \{(1, \hat{y}) \mid y \in x\}$$

and verify that $\mathrm{val}_{\mathcal{G}}(\hat{x}) = x$, for $x \in M$.

1·0 THEOREM *There is a supertransitive model* $\mathbf{K}$ *of* Z, *with* $\omega \in \mathbf{K} \subset V_{\omega+\omega}$ *and* $\mathbf{K}[\mathcal{G}_t] = \mathbf{HF}$.

Proof : Let $K_0 = \omega$, $K_{n+1} = \mathcal{P}(K_n)$ and $\mathbf{K} = \bigcup_{n\in\omega} K_n$. This is a model mentioned by Moschovakis and Enderton; ω is a member of it but $\mathbf{HF}$ is not. To see that $\mathbf{K}[\mathcal{G}] = \mathbf{HF}$, we note that $x \subseteq K_0 \implies \mathrm{val}_{\mathcal{G}}(x) = \varnothing$ and that $(p, y) \in x \subseteq K_{3n+3} \implies y \subseteq K_{3n}$, from which it follows that

$$x \subseteq K_{3n} \implies \varrho(\mathrm{val}_{\mathcal{G}}(x)) \leqslant n. \qquad \dashv (1{\cdot}0)$$

1·1 REMARK For each $x \in V$, the transitive closure of $\hat{x}$ contains only finitely many ordinals.

1·2 DEFINITION For any limit ordinal λ, we define $\mathbf{M}_{13,\lambda} =_{\mathrm{df}} \bigcup\{u \mid \bigcup u \subseteq u \;\&\; u \cap \lambda < \lambda\}$.

It was shown in [M2, section 7] that $\mathbf{M}_{13,\lambda}$ is a supertransitive proper class, with $\mathbf{M}_{13,\lambda} \cap On = \lambda$, and is a model of Z; by the Remark, each $\hat{x} \in \mathbf{M}_{13,\lambda}$. Hence $\mathbf{M}_{13,\lambda}[\mathcal{G}_t] = V$.

1·3 THEOREM *There is a supertransitive model* $\mathbf{N}$ *of* Z, *a proper subset of* $V_{\omega+\omega}$, *such that* $\mathbf{N}[\mathcal{G}_t] = V_{\omega+\omega}$.

Proof : Take $\mathbf{N} = \bigcup\{u \mid \bigcup u \subseteq u \;\&\; \varrho(u) < \omega + \omega \;\&\; u \cap \omega < \omega\}$, which equals $\mathbf{M}_{13,\omega} \cap V_{\omega+\omega}$, then it is readily checked that $\mathbf{N}$ is of height $\omega + \omega$ and $\mathbf{N}[\mathcal{G}_t] = V_{\omega+\omega}$. $\dashv$ (1·3)

1·4 Suppose that $F : On \longrightarrow V$ is a function such that for each ζ, $F(\zeta) \in F(\zeta{+}1) \subseteq \mathcal{P}(F(\zeta))$, that $F(0) = \varnothing$ and that at a limit stage η, $F(\eta) = \bigcup_{\nu<\eta} F(\nu)$; so that each $F(\zeta)$ is transitive.

1·5 DEFINITION For λ a limit ordinal, set

$$A_{F,\lambda} =_{\mathrm{df}} \{u \mid \textstyle\bigcup u \subseteq u \;\&\; \sup\{\xi \in u \cap On \mid F(\xi) \in u\} < \lambda\}; \quad M_{F,\lambda} = \bigcup A_{F,\lambda}.$$

1·6 PROPOSITION $M_{F,\lambda}$ *is a supertransitive model of* Z *for which* $F(\xi) \in M_{F,\lambda} \iff \xi < \lambda$. *If* $F(\zeta)$ *is an ordinal only for finitely many* ζ, *then* $M_{F,\lambda}$ *will contain all ordinals and thus be a proper class.*

Proof : as in Section 7 of [M2]. The union of two members of $A_{F,\lambda}$ is in $A_{F,\lambda}$, and if $u \in A_{F,\lambda}$, so is $\mathcal{P}(u)$; so that $M_{F,\lambda}$ will be a supertransitive model of Z. $\dashv$ (1·6)

1·7 COROLLARY *There is a supertransitive class model* $\mathbf{C}$ *of* Z *which contains a Cohen generic real* c , *and all constructible sets, but such that neither* $L_{\omega+\omega}(c)$ *nor* $P^c_{\omega+\omega}$ *is in* $\mathbf{C}$.

Proof : take $\lambda = \omega + \omega$ and $F(\zeta) = P^c_\zeta$ and $\mathbf{C} = M_{F,\lambda}$. $c \in P^c_\zeta$ whenever $\zeta \geqslant \omega + 1$, so that each $L_\eta \in A_{F,\lambda}$ and $L \subseteq M_{F,\lambda}$. $\dashv$ (1·7)

1·8 REMARK The model $\mathbf{C}$ is the same as Model $\mathbf{M}_{18,\omega+\omega}$ discussed in [M3, section 8], where several constructions based on variants of Proposition 1·6 above are given, to illustrate ways in which rudimentary recursions might fail in models of Z.

Discussion

Evidently the cause of the pathology in the first two examples is that $\mathbf{N}$ contains too many hats, whereas $\mathbf{K}$ contains too few.

If M is $\varnothing$-provident, then first, all members of M will have a hat in M; so that $M \subseteq M[\mathcal{G}]$; and secondly for the trivial forcing and generic $\mathcal{G}$ considered above, $\mathcal{G}$ will be in M, and therefore $\mathrm{val}_{\mathcal{G}}(x)$ will be in M for each x in M, $\mathrm{val}_{\mathcal{G}}(\cdot)$ being defined by a rud recursion; so that $M[\mathcal{G}] \subseteq M$; thus $M[\mathcal{G}] = M$, as one would hope. In particular, the only hats in M would be those of members of M,

The model $\mathbf{C}$ shows that the construction of the generic extension, in this case of L by the original Cohen forcing, requires yet another rudimentary recursion that Z cannot support.

2: Forming the lune of a model

The system M_0 may be given as the axioms of *Extensionality, Null Set, Pairing, Union,* and *Power Set,* ("$\mathcal{P}(x) \in V$") plus the scheme of Δ_0 *Separation.* The system M_1 is M_0 + TCo + $\mathsf{Foundation}$. The system M is M_1 + $\mathsf{Infinity}$.

Axiom H as given in [M1] is this statement:

$$\forall x \exists T \big(\bigcup T \subseteq T \;\&\; \forall v(\bigcup v \subseteq v \;\&\; \overline{\overline{v}} \leqslant \overline{\overline{x}} \implies v \subseteq T)\big).$$

There is a variant of that, which we shall call Axiom H^*:

$$\forall x \exists T \big(\bigcup T \subseteq T \;\&\; \forall v(\bigcup v \subseteq v \;\&\; \overline{\overline{v}} \leqslant^* \overline{\overline{x}} \implies v \subseteq T)\big).$$

Here $\mathfrak{m} \leqslant^* \mathfrak{n}$ is Tarski's relation between cardinals which holds iff there is a surjection of a set of size $\mathfrak{n}$ onto a set of size $\mathfrak{m}$.

2·0 Lemma $\mathsf{M} \vdash \mathsf{H} \iff \mathsf{H}^*$.

Proof: If $\mathfrak{m} \leqslant^* \mathfrak{n}$, $2^{\mathfrak{m}} \leqslant 2^{\mathfrak{n}}$. If $0 < \mathfrak{m} \leqslant \mathfrak{n}$ then $\mathfrak{m} \leqslant^* \mathfrak{n}$. ⊣ (2·0)

2·1 Proposition ([M1, 3·2, page 135]) *Over* M_0, H *is equivalent to Mostowski's principle that every extensional well-founded relation is isomorphic to a transitive set.*

Section 2 of [M1] discusses a construction that starting from any model $\mathbf{M}$ of M_0 yields a model, which in this paper we shall call Lune($\mathbf{M}$), of M_1 + H: if, further, $\mathbf{M}$ satisfies M_1, it will be interpretable as a submodel of Lune($\mathbf{M}$).

A similar construction, though differing in various details, is presented in Hinnion's thesis [H] which starting from models of NF yields models of subsystems of ZF.

The passage from $\mathbf{M}$ to Lune($\mathbf{M}$) can be considered in two ways. If $\mathbf{M}$ is (a transitive set and) a β-model in the sense that every binary relation $(a, r) \in \mathbf{M}$ that is considered by $\mathbf{M}$ to be extensional and well-founded is genuinely well-founded, we can form the larger model by taking every such (a, r), transitising it, and taking the union of the transitisations of all such well-founded extensional relations in $\mathbf{M}$.

But if $\mathbf{M}$ is not a β-model, the construction still is useful; we take all extensional well-founded relations in the sense of $\mathbf{M}$, and working within $\mathbf{M}$, define an interpretation $(\cdot)^1$ as in [M1]; externally to $\mathbf{M}$ we can define an equivalence relation and form the set of equivalence classes, and define a "membership" relation on that set.

2·2 Definition Lune($\mathbf{M}$) $=_{\mathrm{df}}$ the resulting model.

2·3 Remark A case where Lune($\mathbf{M}$) is ill-founded is provided by the model $\mathbf{M}$ described near the end of section 6 of *Strength*, on pages 191-193, where we have added a non-standard quantity of Cohen reals to obtain a transitive model with a long well-ordering of the continuum not isomorphic to any standard ordinal. $\mathbf{M}$ is transitive, but in it there is an ill-founded "well-ordering" of its continuum, and therefore the corresponding ordinal of Lune($\mathbf{M}$) is ill-founded.

Some properties of Lune

2·4 Remark If $\mathbf{M}$ is a β-model, Lune($\mathbf{M}$) will be too, as in Lune($\mathbf{M}$) everything is isomorphic to something in $\mathbf{M}$.

2·5 Remark A transitive model of MOST is necessarily a β-model as every well-founded relation is ranked by ordinals and the ordinals are true ordinals.

2·6 Lemma *If* $\mathbf{M} \models \mathsf{TCo}$ *then any subset* c *in* Lune($\mathbf{M}$) *of a member* d *of* $\mathbf{M}$ *is in* $\mathbf{M}$.

Proof: The subset c is represented as (γ, a, r) say: and d as (δ, a, r); let d be in the transitive set u in $\mathbf{M}$. In $\mathbf{M}$ consider the maximum partial isomorphism from $(u, \in_u)$ to (a, r): δ must be in its range; so using γ we can now recover c. ⊣ (2·6)

2·7 Definition If M and N are two models, not necessarily standard ones, of some system of set theory, we shall write $M \subseteq_d N$ to mean that M is a submodel of N with the further property that whenever $a \in M$, $c \in N$, and $N \models c \subseteq a$ then $c \in M$; thus we have just proved that provided $\mathbf{M} \models \mathsf{TCo}$, $\mathbf{M} \subseteq_d$ Lune($\mathbf{M}$).

2·8 Lemma *Suppose that M, N, a, b, f are such that*

(i) M is a submodel of N; (ii) $N \models \mathsf{GJ}_0$;

(iii) $b \in N$, $a \in M$, $f \in N$, and $N \models f : a \overset{\text{onto}}{\longrightarrow} b$;

(iv) for each $c \in N$ with $N \models c \subseteq a$, $c \in M$;

(v) $M \models \mathcal{P}(a)$ exists.

Then $N \models \mathcal{P}(b)$ exists.

Proof : Let $N \models g = f^{-1}\text{“} \wedge h = g\text{“}$. Such g and h exist in N by (ii). If $N \models d \subseteq b$, then by (iii) $g(d) \subseteq a$, and so by (iv) $g(d)$ is in M; further $h(g(d)) = d$. By (v), let $M \models x = \mathcal{P}(a)$; by (i), $x \in N$; let $y \in N$ with $N \models y = h\text{“}x$. Then $N \models y = \mathcal{P}(b)$. ⊣ (2·8)

Proof in $\mathsf{M} + \mathsf{H}$ that each P^c_ν exists

2·9 Definition We write $\mathcal{S}(x)$ for the set of finite subsets of x; for ordinal arguments, $\mathcal{S}$ is definable by a rudimentary recursion:

$$\mathcal{S}(0) = \{\varnothing\}; \qquad \mathcal{S}(\nu + 1) = \mathcal{S}(\nu) \cup \{x \cup \{\nu\} \mid x \in \mathcal{S}(\nu)\}; \qquad \mathcal{S}(\lambda) = \bigcup_{\nu<\lambda} \mathcal{S}(\nu).$$

2·10 Definition For a given transitive set c let C be the class of finite sequences of elements of $\mathcal{S}(\omega) \cup c$.

2·11 Remark In M one can easily prove that C is a set using the axioms of infinity and power set.

We are about to present an apparently circular argument. In fact what we are proving is that if for each $\nu < \lambda$, a limit ordinal, the set P^c_ν exists, then $\bigcup_{\nu<\lambda} P^c_\nu$ is a set and thus P^c_λ exists.

2·12 Proposition *For any ordinal $\nu \geqslant \omega$, if P^c_ν exists, then there is a surjection $f_\nu : C \times \mathcal{S}(\nu) \overset{\text{onto}}{\longrightarrow} P^c_\nu$.*

Remark In fact one could prove the proposition as stated for all $\nu > 0$, but it hardly seems worth starting earlier than stage ω.

Proof : (initial step) if $\nu = \omega$, then, using the Ackermann relation E on the integers such that $(\omega, E) \cong \mathbf{HF}$, one may easily define f_ω.

(successor step) Suppose $\omega \leqslant \zeta$ and that we have defined $f_\zeta : C \times \mathcal{S}(\zeta) \overset{\text{onto}}{\longrightarrow} P^c_\zeta$. Let $\nu = \zeta + 1$: then $P^c_\nu = \mathbb{T}(P^c_\zeta) \cup c_\nu \cup \{c_\zeta\}$.

We list in remark 2·74 of *Weak Systems* twelve ternary rudimentary functions $S_i(\cdot\,;\cdot,\cdot)$ $(i < 12)$ such that for any non-empty transitive u,

$$\mathbb{T}(u) = \bigcup_{i<12} S_i\text{“}\big(\{u\} \times (u \times u)\big).$$

So if $z \in \mathbb{T}(P^c_\zeta)$ then there are x and y in P^c_ζ, and $i < 12$ with $z = S_i(P^c_\zeta; x, y)$. In turn there are γ and δ in C and s and t in $\mathcal{S}(\zeta)$ with $x = f_\zeta(\gamma, s)$ and $y = f_\zeta(\delta, t)$. Let $u = s \cup t$, and let p, q, r in $\mathcal{S}(\omega)$ be such that $r = p \cup q$, $s = u^{p,r}$ and $t = u^{q,r}$: that notation means that we choose r with $|u| = |r|$, and, writing $\pi_{r,u}$ for the unique order-preserving bijection between r and u (which exists as they are sets of ordinals of the same order-type), we choose p and q so that $s = \pi_{r,u}\text{“}p$ and $t = \pi_{r,u}\text{“}q$

Let $n = \ell h(\gamma)$ and let $\varepsilon = \langle i, p, q, r, n\rangle {}^\frown \gamma {}^\frown \delta$.

Then $u \in \mathcal{S}(\zeta)$ and $\varepsilon \in C$; and we may define f_ν so that in the present case, putting $v = u \cup \{\zeta\}$, $f_\nu(\varepsilon, v) = z$, as follows.

In the present, favourable, case, $\varepsilon(0)$ is a natural number less than 12; $\varepsilon(1)$, $\varepsilon(2)$, and $\varepsilon(3)$ are finite subsets of ω with the third equalling the union of the first two: $\zeta \in v$ and $|v \smallsetminus \{\zeta\}| = |\varepsilon(3)|$; $\varepsilon(4)$ is a natural number; and the length $\ell h(\varepsilon)$ is at least $5 + \varepsilon(4)$.

We may recover γ and δ, which we now call γ_ε and δ_ε by the specifications

$$\begin{gathered}\gamma_\varepsilon \in C; \qquad \ell h(\gamma_\varepsilon) = \varepsilon(4); \qquad \forall k{:<}\varepsilon(4)\ \gamma_\varepsilon(k) = \varepsilon(5 + k);\\ \delta_\varepsilon \in C; \qquad \ell h(\delta_\varepsilon) = \ell h(\varepsilon) - 5 - \varepsilon(4); \qquad \forall k{:<}\ell h(\delta_\varepsilon)\ \delta_\varepsilon(k) = \varepsilon(5 + \varepsilon(4) + k).\end{gathered}$$

Our goal will thus be achieved if we define for $\varepsilon \in C$ and $u \in \mathcal{S}(\zeta)$,

$$\begin{aligned}
f_\nu(\varepsilon, u) &= f_\zeta(\varepsilon, u);\\
f_\nu(\varepsilon, u \cup \{\zeta\}) &= S_{\varepsilon(0)}\big(P^c_\zeta; f_\zeta(\gamma_\varepsilon, u^{\varepsilon(1),\varepsilon(3)}), f_\zeta(\delta_\varepsilon, u^{\varepsilon(2),\varepsilon(3)})\big) \text{ if } \varepsilon \text{ is favourable};\\
f_\nu(\langle 12, \{x\}\rangle, \{\zeta\}) &= \begin{cases} x & \text{if } x \in c_\nu,\\ \varnothing & \text{otherwise;}\end{cases}\\
f_\nu(\langle 13\rangle, \{\zeta\}) &= c_\zeta;\\
f_\nu(\varepsilon, u \cup \{\zeta\}) &= \varnothing \text{ in all other cases.}
\end{aligned}$$

(limit step) Suppose that λ is a limit ordinal $> \omega$ and that for all ν with $\omega \leqslant \nu < \lambda$ we have defined f_ν. For non-empty $t \in \mathcal{S}(\lambda)$, let $t' = t \cap \bigcup t$, so that t is the disjoint union of t' and $\max t$.

Define for $\gamma \in C$ and $t \in \mathcal{S}(\lambda)$,

$$f_\lambda(\gamma, t) = \begin{cases} f_{\max t}(\gamma, t') & \text{if } t \neq \varnothing \text{ and } \max t \geqslant \omega;\\ \varnothing & \text{otherwise.}\end{cases}$$

⊣ (2·12)

2·13 Remark The function $\nu \mapsto f_\nu$ is easily seen to be given by a rudimentary recursion in the parameters c, C, and G_+, where the latter is the graph of addition of natural numbers. As G_+ is a member of every provident set other than **HF**, our definitions would progress happily in any provident set of which c and ω are members.

2·14 Remark If c were empty, or more generally came equipped with a well-ordering, we would be near to the ideas of Koepke and van Eijmeren [vE, K1,2]: they allow Skolem functions to grow as well as well-orderings; in our context c might well be a set of reals with no natural well-ordering, so that definable Skolem functions might not exist.

2·15 Remark Axiom H proves the existence of **HF**: take T including all countably infinite transitive sets.

2·16 Proposition $(\mathsf{M} + \mathsf{H})$ PROV.

Proof: the first thing to check is that H guarantees the existence of $\aleph(a)$ and $\aleph^*(a)$ for each set a.

Then one knows that TCo is provable; whence transitive closures exist, using the power set axiom and restricted separation.

Now to show that ranks exist, let u be a transitive set, and let $\kappa = \aleph^*(\mathcal{P}(u))$. Then we may run the usual recursion but into κ, since each positive ordinal generated will be a surjective image of $\mathcal{P}(u)$.

To prove that each P^c_ν exists we use the above lemma, and the variant H^*. In effect we are imitating the proof of Proposition 4.19 of [M1]. ⊣ (2·16)

3: The provident closure of a set

If M is a transitive non-empty model of AxPair and of TCo, the *provident closure* of M is readily defined thus:

3·0 Definition $\mathrm{Prov}(M) =_{\mathrm{df}} \bigcup\{P^c_\theta \mid \bigcup c \subseteq c \in M\}$, where θ is the least indecomposable ordinal not less than $\varrho(M)$.

3·1 Remark $\varrho(M)$ will be a limit ordinal. If $\varrho(M)$ is indecomposable, $\theta = \varrho(M)$, otherwise θ will be the ordinal product $\varrho(M)\omega$.

3·2 Proposition *(i)* $M \subseteq \mathit{Prov}(M)$*; (ii)* $\mathit{Prov}(M)$ *is provident;*

(iii) if P *is provident and* $M \subseteq P$*, then* $\mathit{Prov}(M) \subseteq P$.

Proof: (i) holds as M models TCo; (ii) holds by the results of [M3, section 5]: specifically, Theorem 5·42 states that P^c_θ is provident whenever θ is indecomposable and c is transitive; Proposition 5·48 implies that an upwards-directed limit of provident sets is provident; and as M models AxPair, $\{c \in M \mid \bigcup c \subseteq c\}$ will be upwards directed; (iii) holds since if $\bigcup c \subseteq c$, $\nu \mapsto P^c_\nu$ is given by a rudimentary recursion. ⊣ (3·1)

3·2 Example $\mathrm{Prov}(\mathbf{M}_{13,\omega+\omega} \cap V_{\omega^2}) = V_{\omega^2}$, because all the hats of members of V_{ω^2} are in $\mathbf{M}$, from which the members can be recovered by rud rec procedures.

Now we wish to relate $\mathrm{Prov}(\mathbf{M})$ to $\mathrm{Lune}(\mathbf{M})$.

3·3 Proposition *Let* $\mathbf{M}$ *be a transitive model of* M*, and let* $H = \mathrm{Lune}(\mathbf{M})$*, which might be ill-founded. Then the standard part of* H *is provident.*

Proof: By applying Proposition 2·16 within $\mathrm{Lune}(\mathbf{M})$, we know that every p-rudimentary recursion succeeds; in particular, rank is definable within $\mathrm{Lune}(\mathbf{M})$, and the standard part of $\mathrm{Lune}(\mathbf{M})$ are those members whose rank is a standard ordinal. It only remains to remark that the sum of two standard ordinals is standard and that if c is a transitive set in the standard part of H and ν is a standard ordinal, the set P^c_ν, which exists in H, is standard. ⊣ (3·3)

3·4 Remark If AC were true in $\mathbf{M}$, we could alternatively argue that by Theorem 3.13 of [M1], H models KP; by Mlle Ville's lemma, its standard part is admissible; and all admissible sets are provident.

3·5 Proposition *The provident closure of* $\mathbf{M}$ *is a subset of the standard part of* $\mathrm{Lune}(\mathbf{M})$*;*

3·6 Corollary $\mathbf{M} \subseteq_d \mathit{Prov}(\mathbf{M})$.

The next proposition is, perhaps, surprising.

3·7 Theorem *Suppose* $\mathbf{M}$ *is a transitive model of* $\mathsf{Z} + \mathsf{TCo}$ *or of* M*; then so is its provident closure* I.

Proof: The two main problems will be the power set axiom and the full separation scheme. To show that the power set axiom is true in I, the following proposition will, by 2·6, 2·7 and 3·6, suffice:

3·8 Proposition *If* $y \in I$ *then there are* $f \in I$ *and* $x \in M$ *with* $f : x \xrightarrow{\text{onto}} y$.

Proof: The f's will be fragments of rud rec functions. We must first show that if a well-founded relation in $\mathbf{M}$ is isomorphic to one in I then there is an isomorphism in I; in fact we construct these isomorphisms in the reverse direction in order to present them as rudimentary recursions on the membership relation.

3·9 Lemma *Let* $u \in I$ *be transitive and isomorphic to an extensional well-founded relation* (a, r) *in* $\mathbf{M}$*. Then the isomorphism, which is unique, belongs to* I.

Proof: In $\mathbf{M}$, note that if $x \subseteq a$, there is, by extensionality of r, at most one $\alpha \in a$ such that $x = \{\beta \in a \mid \beta \in_r \alpha\}$. Let X be the set of such subsets of (a). $X \subseteq \mathcal{P}(a)$ and is a set by Δ_0 separation. For $x \in X$ let $h(x)$ be the corresponding α Then $h \in \mathbf{M} \subseteq I$.

Now in I define $\varpi : u \longrightarrow a$ by

$$\varpi(x) = h(\varpi\text{“}x).$$

Since $h \in I$, that can be construed as a h-rudimentary recursion; since I is provident, $\varpi \restriction u$ is in I and is the desired isomorphism. ⊣ (3·9)

The Proposition follows from the Lemma, the construction of H and the fact that $\mathbf{M} \subseteq I \subseteq H$. ⊣ (3·8)

Proof of 3·7, continued: To show that if $\mathbf{M}$ models full separation, so does I, we must show that there is an interpretation of I in $\mathbf{M}$: that is nearly obvious, as there is an interpretation of the much larger model H

in $\mathbf{M}$; the problem is that as two models $\mathbf{M}$ might have the same lune, it is unclear that $\mathbf{M}$ will always be definable in H.

3·10 REMARK Indeed if K is the Moschovakis–Enderton model discussed in Section 1, then for any supertransitive E with $K \subseteq E \subseteq V_{\omega+\omega}$, $\mathrm{Lune}(E) = H_{\beth_\omega}$. There are $\beth_{\omega+1}$ such sets. So some of them are not definable over $H_{\beth_\omega}$ even allowing parameters from $H_{\beth_\omega}$ in their definition.

3·11 PROPOSITION $\mathbf{M}$ *is not a member of* Lune($\mathbf{M}$).

Proof : If $\mathbf{M} \in \mathrm{Lune}(\mathbf{M})$, there are $(a, r) \in \mathbf{M}$ and $\psi \in \mathrm{Lune}(\mathbf{M})$ such that $\psi : (a,r) \cong \mathbf{M}$. Set $B = \{x \in a \mid x \notin \psi(x)\}$. Then $B \subseteq a \in \mathbf{M}$; $B \in \mathrm{Lune}(\mathbf{M})$, so $B \in \mathbf{M}$. So for some $b \in a$, $B = \psi(b)$. But then $b \in B \iff b \in \psi(b) \iff b \notin B$, a contradiction. ⊣ (3·11)

3·12 REMARK Once we have proved the main result of §4, we shall be able to improve that to the statement that $\mathbf{M}$ lies in no set-generic extension of Lune($\mathbf{M}$).

So we must look for a more specific interpretation. For H we would consider all (α, a, r) where (a, r) is well-founded and extensional and $\alpha \in a$. For I we look at such (a, r) only when they are constructible as codes of some P^c_ν where c is in $\mathbf{M}$ and ν is a finite sum of ordinals that are ranks of elements of $\mathbf{M}$.

3·13 LEMMA *Each ordinal less than* θ *is the sum of finitely many ordinals less than* $\kappa = \varrho(M)$.

Proof : trivial if $\kappa = \theta$; otherwise κ will be decomposable and so the sum of two ordinals less than κ, and each ordinal less than θ is bounded by some finite multiple of κ. ⊣ (3·13)

3·14 LEMMA *Every ordinal smaller than the rank of a transitive set is the rank of a member of it.*

3·15 REMARK There are in $\mathbf{M}$ transitive sets of every rank, for by Corollary 3·7, if $c \in \mathbf{M}$ is transitive, all the c_ζ's, which are certainly in Prov($\mathbf{M}$), will be in $\mathbf{M}$.

3·16 We seek now to characterise among transitive sets the P^c_λ for limit λ, at least when $\lambda > \varrho(c)$. Consider the following type in the variable $\mathfrak{c}$, namely the conjunction of

(3·16·0) $\mathfrak{c}$ is transitive;
(3·16·1) there is no largest ordinal;
(3·16·2) GJ_0, the finitely axiomatisable theory of which the transitive models are the rud closed sets;
(3·16·3) each set is in the domain of an attempt at the rank function;
(3·16·4) each ordinal is in the domain of an attempt at the sequence $\langle \mathfrak{c}_\nu \mid \nu \in ON\rangle$;
(3·16·5) each ordinal is in the domain of an attempt at the sequence $\langle \mathsf{P}^{\mathfrak{c}}_\nu \mid \nu \in ON\rangle$;
(3·16·6) each set is a member of some $P^{\mathfrak{c}}_\nu$.

We abbreviate that formula as $\mathtt{V} = \mathtt{P}(\mathfrak{c})$.

3·17 PROPOSITION *If u is transitive, $c \in u$ and $u \models \mathtt{V} = \mathtt{P}[c]$, then $u = P^c_{On\cap u}$.*

3·18 PROPOSITION *If (a, r) is well-founded and extensional and models $\bigvee\mathfrak{c}\, \mathtt{V} = \mathtt{P}(\mathfrak{c})$, then for some transitive c and limit $\lambda > \varrho(c)$, $\varpi_R : (a, r) \cong (P^c_\lambda, \in_{P^c_\lambda})$.*

3·19 DEFINITION Let (a, r) be well-founded and extensional and model $\mathtt{V} = \mathtt{P}[c]$ for some $c \in a$. We say that (a, r) is *c-small* if in addition there is a second well-founded and extensional relation (b, s) with $\mathrm{Dom}\,(\pi_{arbs}) = a$, which certifies that all the ordinals in (a, r) are less than the rank of c times ω.

We now define, in $\mathbf{M}$, $\mathcal{I}$ to be the class of well-founded extensional (a, r) such that there is a transitive set $d \in \mathbf{M}$ which is represented in (a, r) by a c such that $(a, r) \models \mathtt{V} = \mathtt{P}[c]$ and (a, r) is c-small.

3·20 PROPOSITION *Let $\mathbf{M}$ be a transitive model of* M, *including* TCo. *Then every element of $\mathcal{I}$ is genuinely well-founded, and the union of the transitisations of the members of $\mathcal{I}$ is Prov(M).*

The proof exploits the fact, an easy corollary of results proved in [M3], that if ξ is an ordinal, and e and f are transitive sets with $e \in P^f_\xi$, then for every limit ordinal λ, $P^e_\lambda \subseteq P^f_{\xi+\lambda}$.

3·21 PROPOSITION *If in addition $\mathbf{M} \models \mathsf{Z} + \mathsf{TCo}$, then Prov($\mathbf{M}$) $\models \mathsf{Z} + \mathsf{TCo} + \mathsf{PROV}$.*

Proof : Following the proof of Lemma 2·37 of [M1] we may translate statements of the form $\mathrm{Prov}(\mathbf{M}) \models \Phi$ to statements inside $\mathbf{M}$ about $\mathcal{I}$, and then use separation inside $\mathbf{M}$ to construct the required representatives. ⊣ (3·21)

The proof of Theorem 3·7 is now complete. ⊣ (3·7)

4: Axioms preserved by forcing

In this section we shall speak of various axioms and systems as being *persistent*: by that we shall mean that if true in the ground model they will remain true in each set-forcing extension. There is a certain vagueness, as we do not specify the rest of the axioms of set theory supposed to hold in the ground model; but in view of the main theorem of [M4] we shall assume that we are doing set forcing over a provident set, and seek to show that further axioms, if added, will be forced. The examples we have in mind are the Power Set axiom, Axiom H and $\mathsf{KP}^{\mathcal{P}}$.

4·0 LEMMA *Let I and H be two provident sets with $I \subseteq_d H$. Let $\mathbb{P} \in I$ and let $\mathcal{G}$ be $(I, \mathbb{P})$ generic. Then $\mathcal{G}$ is $(H, \mathcal{G})$ generic, and $I[\mathcal{G}] \subseteq_d H[\mathcal{G}]$.*

Proof : The first remark follows since every subset of $\mathbf{P}$ in H is in I and therefore is met by $\mathcal{G}$. Suppose that $\Vdash \underline{x} \mathrel{\dot\subseteq} \underline{X}$, where $X \in I$ and $x \in H$. Let

$$A = \{(p,\, y) \mid_{p,y}\, p \Vdash_{1} \underline{y} \mathrel{\epsilon} \underline{X} \;\&\; p \Vdash \underline{y} \mathrel{\epsilon} \underline{x}\} \subseteq \mathbf{P}^2 \times \bigcup{}^2 X$$
$$B = \{(p,\, y) \mid_{p,y}\, p \Vdash_{1} \underline{y} \mathrel{\epsilon} \underline{X}\} = \mathcal{A}_1(X)$$

Then $A \in H$ by rud rec separation, $A \subseteq B \in I$; so $A \in I$, by hypothesis.

We must check that $\mathrm{val}_{\mathcal{G}}(A) = \mathrm{val}_{\mathcal{G}}(x)$; we know that $\mathrm{val}_{\mathcal{G}}(B) = \mathrm{val}_{\mathcal{G}}(X)$.

$\mathrm{val}_{\mathcal{G}}(A) = \{\mathrm{val}_{\mathcal{G}}(y) \mid \exists p \mathbin{:\in} \mathcal{G}\; p \Vdash_{1} \underline{y} \mathrel{\epsilon} \underline{X} \;\&\; p \Vdash \underline{y} \mathrel{\epsilon} \underline{x}\}$. Plainly $\mathrm{val}_{\mathcal{G}}(A) \subseteq \mathrm{val}_{\mathcal{G}}(x)$. If $t \in \mathrm{val}_{\mathcal{G}}(x)$, since $\mathrm{val}_{\mathcal{G}}(x) \subseteq \mathrm{val}_{\mathcal{G}}(X)$, we may take $t = \mathrm{val}_{\mathcal{G}}(\xi)$ where $\xi \in I$ and for some $p \in \mathcal{G}$, $p \Vdash_{1} \underline{\xi} \mathrel{\epsilon} \underline{X}$ and $p \Vdash \underline{\xi} \mathrel{\epsilon} \underline{x}$; but then $(p,\, \xi) \in A$ and $\mathrm{val}_{\mathcal{G}}(\xi) \in \mathrm{val}_{\mathcal{G}}(A)$. ⊣ (4·0)

Persistence of the power set axiom

4·1 PROPOSITION $p \Vdash \bigwedge \mathfrak{y}(\underline{\mathfrak{y}} \mathrel{\dot\subseteq} \underline{a} \longrightarrow \Phi(\mathfrak{y})) \iff \forall s_{\subseteq \mathcal{A}_1(a)}\, p \Vdash \Phi[\underline{s}]$.

Proof: Suppose $p \Vdash \bigwedge \mathfrak{y}(\underline{\mathfrak{y}} \mathrel{\dot\subseteq} \underline{a} \longrightarrow \Phi(\mathfrak{y}))$. Let $s \subseteq \mathcal{A}_1(a)$; $\Vdash \underline{s} \mathrel{\dot\subseteq} \underline{a}$; therefore $p \Vdash \Phi[\underline{s}]$. In the other direction, towards a contradiction, suppose that $\forall s_{\subseteq \mathcal{A}_1(a)}\, p \Vdash \Phi[\underline{s}]$, but that $\exists q \mathbin{:\leq} p\; \exists b\; q \Vdash \underline{b} \mathrel{\dot\subseteq} \underline{a} \;\&\; q \Vdash \neg\Phi[b]$. Let $t = a \cap^{\mathbb{P}} b$. Then $t \subseteq \mathcal{A}_1(a)$, so that $p \Vdash \Phi[\underline{t}]$, and $q \Vdash \underline{t} = \underline{b}$, which is absurd. ⊣ (4·1)

Thus if we define

4·2 DEFINITION $\mathcal{P}^{\mathbb{P}}(a) \;=_{\mathrm{df}}\; \{\mathbf{1}^{\mathbf{P}}\} \times \mathcal{P}(\mathcal{A}_1(a))$.
then the power set axiom will assure its set-hood and we may prove the following:

4·3 PROPOSITION $\Vdash \underline{\mathcal{P}^{\mathbb{P}}(a)} = \dot{\mathcal{P}}(\underline{a})$.

Persistence of $\mathcal{S}(x) \in V$

In [M2] we study at some length the operator $\mathcal{S}(x)$, defined as the class of finite subsets of x; in models of certain weak systems, that may for certain x fail to be a set.

4·4 DEFINITION $\mathcal{S}^{\mathbb{P}}(a) \;=_{\mathrm{df}}\; \{\mathbf{1}^{\mathbf{P}}\} \times \mathcal{S}(\mathcal{A}_1(a))$.
If $\mathcal{S}(x) \in V$ holds, then $\mathcal{S}^{\mathbb{P}}(a)$ will be a set, and as above we may prove the following:

4·5 PROPOSITION $\Vdash \underline{\mathcal{S}^{\mathbb{P}}(a)} = \dot{\mathcal{S}}(\underline{a})$.

Persistence of flat restricted collection

4·6 PROPOSITION $\mathsf{FlatColl}$ *is persistent.*

Proof : Suppose that $p \Vdash \bigwedge \mathfrak{x} \mathbin{:\epsilon} \underline{a} \bigvee \underline{y} \mathrel{\dot\subseteq} \underline{d}\; \dot\Phi$.

Then

$$\forall q \mathbin{:\leq} p\big[q \Vdash_{1} x \mathrel{\epsilon} \underline{a} \implies \exists y_{\subseteq \mathbf{P} \times \bigcup^2 d}\, \exists r \mathbin{:\leq} q\; r \Vdash \dot\Phi.\big]$$

So there is a B such that $\forall q \mathbin{:\leq} p\big[q \Vdash_{1} \underline{x} \mathrel{\epsilon} \underline{a} \implies \exists y \mathbin{:\in} B\; \exists r \mathbin{:\leq} q\; r \Vdash \Phi(\underline{x}, \underline{y}).\big]$

So form $b \;=_{\mathrm{df}}\; \{(\mathbf{1}^{\mathbf{P}},\, y) \mid y \in B\}$. Then

$$p \Vdash \bigwedge \mathfrak{x} \mathbin{:\epsilon} \underline{a} \bigvee \mathfrak{y} \mathbin{:\epsilon} \underline{b}\, \Phi(\mathfrak{x}, \mathfrak{y}) \;\&\; \mathfrak{y} \mathrel{\dot\subseteq} \underline{d}. \qquad ⊣ (4·6)$$

Persistence of $\mathsf{M} + \mathsf{PROV}$

That theory is PROVI + the power set axiom, and thus is persistent by the above and the main result of [M4].

Persistence of Axiom H

We shall show, assuming $\mathsf{M} + \mathsf{H}$, that "every well-founded extensional relation is isomorphic to a transitive set" is forced, and then invoke Proposition 2·1.

4·7 We start from a, R such that

$$\Vdash \underline{a}\ \texttt{is a set and}\ \underline{R}\ \texttt{is a well-founded extensional relation on it.}$$

Let $A = \mathcal{A}_1(a)$. We are going to define a sequence of subsets of A in what is something like a monotonic inductive definition. Axiom H is enough to sustain this definition and to provide enough ordinals for it to work; so we have in effect defined in the generic extension an isomorphism of the said relation with a transitive set. Our model is this recursive definition:

$$\varpi_R(x) = \{\varpi_R(y) \mid yRx\}.$$

4·8 DEFINITION For any $B \subseteq A$ we put

$$\Phi(B) \ =_{\mathrm{df}} \ \{(p,\, x) \in A \mid p \Vdash \neg(\underline{x} \epsilon \underline{B}) \ \&\ p \Vdash \bigwedge \mathfrak{y}\!: \epsilon \underline{A}(\mathfrak{y}\, \underline{R}\, \underline{x} \longrightarrow \mathfrak{y} \epsilon \underline{B})\}.$$

4·9 REMARK

(i) $\Phi(B)$ is the result of applying to A a rud rec separator, and thus is a set.

(ii) Moreover, $\Phi : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ and will be a set, provably in $\mathsf{M} + \mathsf{H}$.

(iii) By Remark 0·19, $\Phi(B)$ is neat in A.

(iv) $\Vdash \underline{\Phi(B)} \mathbin{\dot\cap} \underline{B} = \dot\varnothing$.

4·10 LEMMA $\Phi(B) = \varnothing \Longleftrightarrow A = B$.

4·11 DEFINITION Let $\kappa = \aleph(\mathcal{P}(A))$. For $\nu \leqslant \kappa$ we define subsets A_ν, B_ν of A.
We put $B_0 = \varnothing$ and $A_0 = \Phi(B_0)$;
at successor stages: put $B_{\nu+1} = \{(p,\, x) \in A \mid p \Vdash \underline{x} \epsilon \underline{A_\nu} \mathbin{\dot\cup} \underline{B_\nu}\}$; $A_{\nu+1} = \Phi(B_{\nu+1})$;
at a limit stage λ, put $B_\lambda = \{(s, \alpha) \in A \mid \forall p\!:\leq s\ \exists q\!:\leq p\ \exists \nu\!:< \lambda\ (q,\, \alpha) \in A_\nu\}$, and put $A_\lambda = \Phi(B_\lambda)$.

4·12 LEMMA *For each* ν,

(4·12·0) B_ν *and* A_ν *are sets, and are neat in* A;
(4·12·1) $\Vdash \underline{A_\nu} \mathrel{\dot\subseteq} \underline{a}$;
(4·12·2) $\mathrm{val}_{\mathcal{G}}(A_\nu) = \{x \mid x$ *is an* $\mathrm{val}_{\mathcal{G}}(R)$*-minimal member of* $\mathrm{val}_{\mathcal{G}}(A) \smallsetminus \mathrm{val}_{\mathcal{G}}(B_\nu)\}$;
(4·12·3) *For* $\nu < \zeta \leqslant \theta$, $\Vdash \underline{A_\nu} \mathbin{\dot\cap} \underline{A_\zeta} = \dot\varnothing$.

4·13 REMARK The A_ν's are pairwise disjoint, and therefore cannot be non-empty for all $\nu < \kappa$, else we would have defined an injection of κ into $\mathcal{P}(A)$, contrary to the choice of κ.

So let θ be the first ordinal with $A_\theta = \varnothing$; then $B_\theta = A$, and for all larger ν, $B_\nu = A$ and $A_\nu = \varnothing$.

4·14 REMARK We may regard the above as a rud rec definition in the parameters Φ, A and R.

4·15 LEMMA $\forall \xi\!:\leq \theta\ \ \mathrm{val}_{\mathcal{G}}(B_\xi) = \bigcup_{\nu<\xi} \mathrm{val}_{\mathcal{G}}(A_\nu)$.

4·16 We define a map $\varpi_R^{\mathbb{P}}$ with domain $\{x \mid \exists p\ (p,\, x) \in B_\theta\}$. We set for such x

$$\varpi_R^{\mathbb{P}}(x) \ =_{\mathrm{df}} \ \bigcup_{\nu<\theta} \{(p,\, \varpi_R^{\mathbb{P}}(y)) \mid (p,\, x) \in A_\nu \ \&\ (p,\, y) \in B_\nu \ \&\ p \Vdash \underline{y}\, \underline{R}\, \underline{x}\}$$

We remark that:

(i) if f and g are the maps $\nu \mapsto A_\nu$ and $\nu \mapsto B_\nu$ for $\nu \leqslant \theta$, then f, g are sets, using the power set axiom, and so the definition of $\varpi_R^{\mathbb{P}}$ may be construed as a rudimentary recursion in the parameters f, g, A and R.

(ii) each $(p,\, x) \in B_\theta$ is in at most one A_ν as they are disjoint as sets; but more than that, if $(p,\, x) \in A_\nu$ and $(q,\, x) \in A_\zeta$ where $\zeta < \nu$, then p and q are incompatible, and so cannot both be in $\mathcal{G}$; since $\Vdash \underline{A_\nu} \mathbin{\dot\cap} \underline{A_\zeta} = \dot\varnothing$.

(iii) for every $(p,\, x) \in B_\theta$ with $p \in \mathcal{G}$, there is a $\nu < \theta$ and a $q \in \mathcal{G}$ with $(q,\, x) \in A_\nu$: for suppose that $p \in \mathcal{G}$; then $p \Vdash \underline{x} \mathrel{\epsilon} \underline{B_\theta}$, because $\Vdash \underline{A} = \underline{B_\theta}$. So $\exists q{:}{\leq}p\, \exists \nu{:}{<}\theta\; q \Vdash \underline{x} \mathrel{\epsilon} \underline{A_\nu}$ so $(q,\, x) \in A_\nu$, since A_ν is neat in A.

Thus

$$\mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(x)) = \bigcup_{\nu<\theta} \{\mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(y)) \big| \exists p{:}{\in}\mathcal{G}\ (p,\, x) \in A_\nu \ \&\ (p,\, y) \in B_\nu \ \&\ p \Vdash \underline{y}\, \underline{R}\, \underline{x}\}$$

Let $\nu(x)$ be the unique ν concerned (which exists by (i) and (ii)). Then

$$\begin{aligned}\mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(x)) &= \big\{\mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(y)) \big| \exists p{:}{\in}\mathcal{G}\ (p,\, x) \in A_{\nu(x)} \ \&\ (p,\, y) \in B_{\nu(x)} \ \&\ p \Vdash \underline{y}\, \underline{R}\, \underline{x}\big\}\\ &\subseteq \big\{\mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(y)) \big| \mathrm{val}_{\mathcal{G}}(x) \in \mathrm{val}_{\mathcal{G}}(A_{\nu(x)}) \ \&\ \mathrm{val}_{\mathcal{G}}(y) \in \mathrm{val}_{\mathcal{G}}(B_{\nu(x)}) \ \&\ \mathrm{val}_{\mathcal{G}}(y)\, \mathrm{val}_{\mathcal{G}}(R)\, \mathrm{val}_{\mathcal{G}}(x)\big\}\\ &\subseteq \big\{\mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(y)) \big| \mathrm{val}_{\mathcal{G}}(y) \in \mathrm{val}_{\mathcal{G}}(B_{\nu(x)}) \ \&\ \mathrm{val}_{\mathcal{G}}(y)\, \mathrm{val}_{\mathcal{G}}(R)\, \mathrm{val}_{\mathcal{G}}(x)\big\}\\ &\subseteq \big\{\mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(y)) \big| \mathrm{val}_{\mathcal{G}}(y)\, \mathrm{val}_{\mathcal{G}}(R)\, \mathrm{val}_{\mathcal{G}}(x)\big\}\end{aligned}$$

Now suppose, for the same x, that in $M[\mathcal{G}]$, $\mathrm{val}_{\mathcal{G}}(y)\, \mathrm{val}_{\mathcal{G}}(R)\, \mathrm{val}_{\mathcal{G}}(x)$; then for some $q \in \mathcal{G}$, $q \Vdash \underline{y}\, \underline{R}\, \underline{x}$; we may suppose there is an $r \in \mathcal{G}$ with $r \Vdash_1 \underline{x} \mathrel{\epsilon} \underline{A_{\nu(x)}}$; so if $p' \in \mathcal{G}$ is below both q and r, $p' \Vdash \underline{y} \mathrel{\epsilon} \overline{B_{\nu(x)}}$; and we can find y' and strengthen p' to a $p \in \mathcal{G}$ with $p \Vdash \underline{y} = \underline{y'}$ and $p \Vdash_1 \underline{y'} \mathrel{\epsilon} \underline{B_{\nu(x)}}$ and $p \Vdash_1 \underline{x} \mathrel{\epsilon} \underline{A_{\nu(x)}}$. In these circumstances, $\mathrm{val}_{\mathcal{G}}(y') = \mathrm{val}_{\mathcal{G}}(y)$; but that, together with the neatness of each A_ν and B_ν, implies that $\mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(y')) = \mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(y))$. We conclude that

$$\mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(x)) = \big\{\mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(y)) \big| \mathrm{val}_{\mathcal{G}}(y)\, \mathrm{val}_{\mathcal{G}}(R)\, \mathrm{val}_{\mathcal{G}}(x)\big\}$$

We now introduce a name, F:

$$F \ =_{\mathrm{df}}\ \{(p,\, (\varpi^{\mathbb{P}}_R(x), x)^{\mathbb{P}}_2) \mid (p,\, x) \in A\}.$$

Here $(a,\, b)^{\mathbb{P}}_2$ is the forcing name defined in [M4] such that $\mathrm{val}_{\mathcal{G}}((a,\, b)^{\mathbb{P}}_2) = (\mathrm{val}_{\mathcal{G}}(a),\, \mathrm{val}_{\mathcal{G}}(b))$. F is a set, using some rud recursion, which is available assuming H.

4·17 PROPOSITION *In* $M[\mathcal{G}]$, $\mathrm{val}_{\mathcal{G}}(F)$ *is a function and for each* $(p,\, x) \in A$,

$$p \in \mathcal{G} \implies (\mathrm{val}_{\mathcal{G}}(F))(\mathrm{val}_{\mathcal{G}}(x)) = \mathrm{val}_{\mathcal{G}}(\varpi^{\mathbb{P}}_R(x))$$

A name for the image of $\mathrm{val}_{\mathcal{G}}(F)$ is easily built: set $u \ =_{\mathrm{df}}\ \{(s,\, \varpi^{\mathbb{P}}_R(\alpha)) \mid (s,\, \alpha) \in A\}$.

4·18 LEMMA $\mathrm{val}_{\mathcal{G}}(u)$ *is transitive and is the image of* $\mathrm{val}_{\mathcal{G}}(F)$.

The persistence of Axiom H has now been proved.

4·19 PROPOSITION *Let* $\mathbf{M} \models \mathsf{M}$. *Then* $\mathbf{M}$ *lies in no set-generic extension of* Lune($\mathbf{M}$).

Proof: Write H for Lune($\mathbf{M}$). Suppose that $\mathbf{M} \in H[\mathcal{G}]$. We now know that $H[\mathcal{G}] \models \mathsf{M} + \mathsf{H}$; so, working in $H[\mathcal{G}]$, we may deduce that there is a transitive set T which includes all transitive sets v with $\overline{\overline{v}} \leqslant \overline{\overline{\mathbf{M}}}$. Among those are transitisations of all well-founded extensional relations that are members of $\mathbf{M}$; so $H \subseteq T \in H[\mathcal{G}]$. Let $T = \mathrm{val}_{\mathcal{G}}(t)$ where $t \in H$. Work now in H, and consider the map $\psi : H \longrightarrow \mathcal{P}(\mathcal{A}_1(t))$ given by $h \mapsto \{(p,\, \tau) \mid p \Vdash \underline{\hat{h}} = \underline{\tau}\}$. It is routine to verify that for $h \in H$, $\psi(h)$ is non-empty, and that indeed ψ is injective. But that is absurd, as ψ is definable over H, the power set axiom is true there, and by Cantor there can be no injection of $\mathcal{P}(\mathcal{P}(\mathcal{A}_1(t))$ into $\mathcal{P}(\mathcal{A}_1(t))$. ⊣ (4·19)

Persistence of $\mathsf{KP}^{\mathcal{P}}$

Much as the persistence of KP was done in [M4], with, in addition, much use of Proposition 4·1.

5: Forcing over provident models of Zermelo set theory

In this section we discuss the possibility of the following diagram commuting:

$$\begin{tikzcd}
 & H \arrow[r, "\mathrm{val}_{\mathcal{G}}"'] & H[\mathcal{G}] \\
\mathbb{P} \in & I \arrow[u, "\mathrm{Lune}"'] \arrow[r, "\mathrm{val}_{\mathcal{G}}"'] & I[\mathcal{G}] \arrow[u, "\mathrm{Lune}"']
\end{tikzcd}$$

5·0 Theorem *Let I and H be transitive sets such that I is a provident model of Z or at least M, and $H = \mathrm{Lune}(I)$. Let $\mathbb{P} \in I$ and let $\mathcal{G}$ be $(I, \mathbb{P})$ generic. Then $\mathcal{G}$ is $(H, \mathbb{P})$ generic, and further $H[\mathcal{G}] = \mathrm{Lune}(I[\mathcal{G}])$.*

Proof : That $\mathcal{G}$ is $(H, \mathbb{P})$ generic is immediate from the fact that $\mathbf{P} \in I$ and so every subset of it which is in H is in I. The analysis of the relationship of $H[\mathcal{G}]$ to $I[\mathcal{G}]$ rests on the commutativity of diagrams of the following kind:

$$\begin{tikzcd}
(A, \in_A) \arrow[r, "\mathrm{val}_{\mathcal{G}}"'] & (A[\mathcal{G}], \in_{A[\mathcal{G}]}) \\
(M, R) \arrow[u, "\varpi_R"] \arrow[r, "\mathrm{val}_{\mathcal{E}}"'] & (N, S) \arrow[u, "\varpi_S"]
\end{tikzcd}$$

which must now be explained.

In the above diagram we suppose that A is a provident member of H which contains $\mathbb{P} \cup \mathcal{P}(\mathbb{P})^I$; so, necessarily, $\chi_=$ and χ_ϵ and ϱ are total on A, so that its rank is the least ordinal not in it; further A will contain the hats of all its members and $\mathcal{G}$ will be $(A, \mathbb{P})$-generic; and so by our observations in Section 1, $A \subseteq A[\mathcal{G}]$. Because H is closed under ordinal multiplication and unordered pairing, it will be a union of provident sets of that kind.

Thus $A[\mathcal{G}]$ will be transitive and a member of $H[\mathcal{G}]$, as $A[\mathcal{G}] = \mathrm{val}_{\mathcal{G}}(B)$, where $B = \{(\mathbf{1}^{\mathbb{P}}, \hat{x}) \mid x \in A\}$.

We wish to show that $A[\mathcal{G}]$ is isomorphic to something in $I[\mathcal{G}]$. That will suffice, as every member of H will be a member of some such A, and so every member of $H[\mathcal{G}]$ will be a member of some such $A[\mathcal{G}]$.

We know that A with the membership relation $\in_A$ ($= \in \cap (a \times a)$)will be isomorphic to some well-founded extensional relation, (M, R) say, in I; we write ϖ_R for the isomorphism from (M, R) to $(A, \in_A)$. $\varpi_R \in H$ since (M, R) is well-founded in H and $H \models \mathsf{MOST}$.

Since $\mathbb{P} \in A$, $\mathbb{P}$ will be isomorphic to something in M. $\mathcal{G}$ is thus isomorphic to a generic $\mathcal{F}$ for (M, R), and we have to define $\mathrm{val}_{\mathcal{F}}$. Let $P \in M$ be given by $\varpi_R(P) = \mathbf{P}$. Then $\mathcal{F} = \{p \in M \mid \varpi_R(p) \in \mathcal{G}\}$. $\mathcal{F} \in H[\mathcal{G}]$ since both $\mathcal{G}$ and ϖ_R are, but it is a subset of $M \in I[\mathcal{G}]$, and so $\mathcal{F} \in I[\mathcal{G}]$ by Lemma 4·6.

Now work in $I[\mathcal{G}]$, which is a model of $\mathsf{M} + \mathsf{PROV}$ since I is. We follow the approach to forcing over arbitrary models of sufficient set theory described in [M1] section 6.

We first define an equivalence relation: for x and y in M, we say

$$x \sim_{\mathcal{F}} y \iff_{\mathrm{df}} \exists p {:}{\in} \mathcal{F} \;\; M \models (p \Vdash \underline{x} = \underline{y}).$$

We write $(x)_{\mathcal{F}}$ for the equivalence class of $x \in M$, which will be a member of $I[\mathcal{G}]$ by $\dot{\Delta}_0$ separation with M and $\mathcal{F}$ as parameters; (we need not here define $M \models \varphi$ in general as we need it only for finitely many instances); and then using the power set axiom in $I[\mathcal{G}]$ we form the collection N of equivalence classes; so $N \in I[\mathcal{G}]$.

Then using $\mathcal{F}$ we define the relation S between two members of N:

$$(y)_{\mathcal{F}} \, S \, (x)_{\mathcal{F}} \iff_{\mathrm{df}} \exists p {:}{\in} \mathcal{F} \;\; M \models (p \Vdash \underline{y} \,\epsilon\, \underline{x}).$$

$S \subseteq N \times N$, so will be a member of $I[\mathcal{G}]$, again by $\dot{\Delta}_0$ separation with M and $\mathcal{F}$ as parameters.

Thus (N, S) is a member of $I[\mathcal{G}]$. We must show that it can be interpreted as a generic extension of (M, R): once we have done that, it will rapidly follow that (N, S) is well-founded, for $(M, R) \models \mathsf{PROV}$ and so set forcing over it adds no new ordinals; and so the ordinals of (N, S) are those of (M, R), which in turn are isomorphic to the ordinals of A which are real ordinals and thus well-founded; but PROV proves the existence of the rank function, and therefore the well-foundedness of the ordinals of (M, R) implies the well-foundedness of the model itself.

To represent (N, S) as a generic extension, we note first that $\mathcal{F}$ is represented in (N, S):

(i) there is an $\dot{\mathcal{F}}$ in M with $(M, R) \models \big[\underline{\dot{\mathcal{F}}} = \{(\mathfrak{p}, \hat{\mathfrak{p}} \mid \mathfrak{p} \epsilon \underline{P}\}\big]$, where $\varpi_R(P) = \mathbf{P}$;

(ii) there is an embedding $i : (M, R) \xrightarrow{1-1} (N, S)$ given by $i(m) = (\hat{m})_{\mathcal{F}}$, where the hat is computed in M; so $m \, R \, n \implies i(m) \, S \, i(n)$, et cetera;

(iii) $i(p) \, S \, (\dot{\mathcal{F}})_{\mathcal{F}} \iff p \in \mathcal{F}$.

Set $\mathcal{E} =_{\mathrm{df}} (\dot{\mathcal{F}})_{\mathcal{F}}$. $\mathcal{E} \in (N, S)$. In (N, S) we may define $\mathrm{val}_{\mathcal{E}}(\cdot)$ by the customary recursion and verify that for all x in M, $\mathrm{val}_{\mathcal{E}}(x) = (x)_{\mathcal{F}}$.

Finally we must verify that when we transitise (N, S) using ϖ_S, we obtain $(A[\mathcal{G}], \in_{A[\mathcal{G}]})$. But ϖ_R and ϖ_S are isomorphisms, and so that will follow by comparing the definitions of $\mathrm{val}_{\mathcal{G}}$ and $\mathrm{val}_{\mathcal{E}}$, as indicated by the following remarks.

5·1 For $x \in M$, $\mathrm{val}_{\mathcal{E}}(x) = (x)_{\mathcal{F}}$; note that

$$\begin{aligned} \varpi_S((x)_{\mathcal{F}}) &= \big\{\varpi_S((y)_{\mathcal{F}}) \bigm| \exists p{:}{\in}\mathcal{F} \ (M, R) \models [p \Vdash \underline{y} \epsilon \underline{x}]\big\} \\ &= \big\{\varpi_S((y)_{\mathcal{F}}) \bigm| \exists p{:}{\in}\mathcal{F} \ (M, R) \models [p \Vdash_1 \underline{y} \epsilon \underline{x}]\big\} \\ \text{whereas} \qquad \mathrm{val}_{\mathcal{G}}(\varpi_R(x)) &= \big\{\mathrm{val}_{\mathcal{G}}(\varpi_R(y)) \bigm| \exists p{:}{\in}\mathcal{G} \ (A, \in_A) \models [p \Vdash \underline{\varpi_R(y)} \epsilon \underline{\varpi_R(x)}]\big\} \\ &= \big\{\mathrm{val}_{\mathcal{G}}(\varpi_R(y)) \bigm| \exists p{:}{\in}\mathcal{G} \ (A, \in_A) \models [p \Vdash_1 \underline{\varpi_R(y)} \epsilon \underline{\varpi_R(x)}]\big\} \end{aligned}$$

but $\mathcal{F} = \{p \in M \mid \varpi_R(p) \in \mathcal{G}\}$.

A transfinite recursion on R will now show that for each $x \in M$, if $\varpi_S(\mathrm{val}_{\mathcal{E}}(y)) = \mathrm{val}_{\mathcal{G}}(\varpi_R(y))$ for each y with yRx, then $\varpi_S(\mathrm{val}_{\mathcal{E}}(x)) = \mathrm{val}_{\mathcal{G}}(\varpi_R(x))$. ⊣ (5·1)

5·2 COROLLARY *Any well founded extensional relation in $I[\mathcal{G}]$ is isomorphic in $H[\mathcal{G}]$ to a transitive set.*

Proof : by Lemma 4·6 it will preserve its well-foundedness in $H[\mathcal{G}]$; and now its collapsibility to a transitive set will hold as we have shown that the truth of Axiom H in H implies the truth of MOST in $H[\mathcal{G}]$. ⊣ (5·2)

5·3 LEMMA *Any transitive set of $H[\mathcal{G}]$ is the collapse of something in $I[\mathcal{G}]$*

Proof : we have essentially already proved that. If u is such a set, it will be $\mathrm{val}_{\mathcal{G}}(x)$ for some $x \in H$; we find a provident $A \in H$ as above with $x \in A$, and then argue as before. ⊣ (5·3)

5·4 REMARK We have proved Theorem 5·0 for the case that I and H are transitive; but the result that there is a commuting diagram will still hold when H or even I fails to be a β-model, as if I is a (countable) non-β possibly ill-founded model of MSF + Z or MAC, we may describe H and $H[\mathcal{G}]$ internally, and translate the proof of Theorem 5·0 accordingly.

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