A SCENARIO FOR TRANSFERRING HIGH SCORES

A. R. D. Mathias

ERMIT, Université de la Réunion

1: Introduction

Let $\mathcal{X}$ be a Polish space with metric d and $f : \mathcal{X} \longrightarrow \mathcal{X}$ a continuous function. We recall the definition of the attacking relation $x \curvearrowright_f y$ studied in our papers [2, 3, 4]:

$$x \curvearrowright_f y \iff_{\mathrm{df}} \forall \varepsilon > 0\ \forall m{:}{\in}\mathbb{N}\ \exists \ell\ [\ell \geqslant m\ \&\ d(f^\ell(x), y) < \varepsilon].$$

Our notation is related to a familiar one in dynamical systems: $x \curvearrowright_f y$ iff $y \in \omega_f(x)$.

An important example is *Baire space*, the space of infinite sequences of natural numbers, often denoted by $\mathcal{N}$ or ${}^\omega\omega$, which for each finite such sequence r has the basic open set $\{\alpha \mid \alpha \restriction \ell h(r) = r\}$. In that space the (backward) *shift function* $\mathfrak{s} : \mathcal{N} \longrightarrow \mathcal{N}$ is given by $\mathfrak{s}(\alpha)(n) = \alpha(n+1)$.

The *score*, $\theta(a, f)$, of a point a in $\mathcal{X}$ with respect to the function f, is defined to be the least ordinal θ such that $A^\theta(a, f) = A^{\theta+1}(a, f)$, where we define recursively a shrinking sequence of sets by $A^0(a, f) = \omega_f(a)$, $A^{\nu+1}(a, f) = \{x \mid \exists y(y \in A^\nu(a, f)\ \&\ y \curvearrowright_f x)\}$ and for a limit ordinal λ, $A^\lambda(a, f) = \bigcap_{\nu<\lambda} A^\nu(a, f)$.

In [3] we showed, working always with the shift function $\mathfrak{s}$, that for Baire space ${}^\omega\omega$ or the Cantor space ${}^\omega 7$ there are points of score any given countable ordinal. In [4] we constructed a point in Baire space of score the first uncountable ordinal, which by results of [3] is the maximum possible. An unpublished transfer theorem of Christian Delhommé shows that a point of uncountable score and points of all countable scores will exist in Cantor space ${}^\omega 2$.

The question arises, which other spaces and functions will permit the existence of a point of uncountable score ?

We describe hypotheses on a dynamical system $(\mathcal{X}, f)$ which will permit such a transfer. We then describe one setting in which one can establish all but the last hypothesis; then we show that an assumption of equicontinuity will yield that last hypothesis. However, experts on dynamical systems believe that, in the given setting, it is likely that f cannot be equicontinuous. So at present no dynamical system is known (to the author) to satisfy all the given hypotheses.

We work with *DC*, the mild form of the Axiom of Choice that implies that a relation is well-founded if and only if it admits no infinite descending sequences.

2: Hypotheses leading to high scores

2·0 THEOREM *Suppose that $(\mathcal{X}, f)$ is a dynamical system such that there is a continuous surjection Ψ of $\mathcal{X}$ onto either ${}^{\omega}\omega$ or some ${}^{\omega}m$ with $m \geqslant 2$, satisfying these properties, where we say that x is at α rather than $\Psi(x) = \alpha$:*

(2·0·0) for all $x \in \mathcal{X}$, $\Psi(f(x)) = \mathfrak{s}(\Psi(x))$

(2·0·1) for all x and y in $\mathcal{X}$, if $x \curvearrowright_f y$ then $\Psi(x) \curvearrowright_{\mathfrak{s}} \Psi(y)$

(2·0·2) if x is at α and $\alpha \curvearrowright_{\mathfrak{s}} \beta$, then there is a y at β with $x \curvearrowright_f y$.

(2·0·3) if $\alpha \curvearrowright_{\mathfrak{s}} \beta \curvearrowright_{\mathfrak{s}} \gamma$, a is at α, c is at γ and $a \curvearrowright_f c$, then there is a point b at β with $a \curvearrowright_f b \curvearrowright_f c$.

Then for every x, the f-score of x equals the $\mathfrak{s}$-score of $\Psi(x)$, so that there are points in $\mathcal{X}$ of all scores up to and including ω_1.

Proof : We recall the definition from [3], page 263, of the tree $T^x_y(f)$, where $x \curvearrowright_f y$. It is the set of finite sequences $(y_0, y_1, \ldots y_n)$ such that $y_0 = y$, each $y_{i+1} \curvearrowright_f y_i$ and $x \curvearrowright_f y_n$.

Extend the definition of Ψ to such finite sequences in the natural way: $\Psi((y_0, \ldots y_n)) = (\Psi(y_0) \ldots \Psi(y_n))$. With this extended definition, Ψ preserves length and end-extension.

2·1 PROPOSITION *If $x \curvearrowright_f y$, $\Psi[T^x_y(f)] = T^{\Psi(x)}_{\Psi(y)}(\mathfrak{s})$.*

Proof : by (2·0·1), the restriction of Ψ to the first tree is into the second; by repeated use of (2·0·3), we see that it is onto. ⊣ (2·1)

2·2 LEMMA *If x is at α, y is at β, and $x \curvearrowright_f y$, then $T^x_y(f)$ is well-founded iff $T^\alpha_\beta(\mathfrak{s})$ is.*

Proof : by repeated use of (2·0·1) and (2·0·3) to transfer infinite descending sequences from one tree to the other. ⊣ (2·2)

2·3 DEFINITION If $a \curvearrowright_f b$, $T^a_b(f)$ is well-founded, and $s \in T^a_b(f)$, write $\varrho_{a,b,f}(s)$ for the rank of s in the tree $T^a_b(f)$, as given by the recursion

$$\varrho_{a,b,f}(s) = \sup\{\varrho_{a,b,f}(s^\frown\langle y\rangle) + 1 \mid s^\frown\langle y\rangle \in T^a_b(f)\}.$$

2·4 PROPOSITION *If $a \curvearrowright_f b$, $\emptyset \neq s \in T^a_b(f)$ and $\varrho_{a,b,f}(s) = \xi$, then the last element $\ell(s)$ of the finite sequence s is in $A^\xi(a, f) \smallsetminus A^{\xi+1}(a, f)$.*

Proof : by induction on ξ, much as in the proof of Lemma 2·1 of [3]. ⊣ (2·4)

2·5 COROLLARY *$\theta(a, f)$, the f-score of a, equals* $\sup\{\varrho_{a,b,f}(\emptyset) \mid a \curvearrowright_f b\}$.

Thus the score of a point a is computable from the ranks of the various trees $T^a_b(f)$.

2·6 PROPOSITION *Suppose the two trees are both well-founded. Then they have the same rank: $\varrho_{a,b,f}(\emptyset) = \varrho_{\alpha,\beta,\mathfrak{s}}(\emptyset)$.*

2·7 PROPOSITION *If x is at α, then the f-score of x equals the $\mathfrak{s}$-score of α; in particular if x is at α and α is of uncountable score, then so is x.*

For points of uncountable score we can also argue as follows. A point is of uncountable score iff its abode is not a Borel set. Further the image under Ψ of the abode of x is the abode of α, and the image under Ψ of the escape of x is the escape of α.

Hence if the abode and escape of x were Borel, then both the abode and the escape of α would be analytic, and therefore by Souslin's celebrated result, would be Borel.

2·8 Remark In this connection it might be worth looking again at the "original" construction of a point of uncountable score mentioned in the penultimate paragraph of [4], and there called c, where c "neatly" attacks α_T for every countable well-founded tree T. The nodes of α_T survive for $\zeta = \varrho(T)$ steps, and ζ can be arbitrarily large.

2·9 Proposition *Suppose that $\alpha \curvearrowright_{\mathfrak{s}} \rho \curvearrowright_{\mathfrak{s}} \rho$. Let a be at α and x be at ρ with $a \curvearrowright_f x$. Then there are recurrent r and s such that $a \curvearrowright_f s \curvearrowright_f x \curvearrowright_f r$ and $\rho \curvearrowright_{\mathfrak{s}} \Psi(r) \curvearrowright_{\mathfrak{s}} \rho$.*

Proof: by (2·0·2) there is a y at ρ such that $x \curvearrowright y$. Taking, by (2·0·3), an infinite backward sequence of ρ's there are points y_i with $y_0 = y$, each $y_{i+1} \curvearrowright_f y_i$ and x attacking each y_i. By proposition 3.18 of [3] there is a recurrent point r with $x \curvearrowright_f r \curvearrowright_f y$, which by (2·0·1) gives $\rho \curvearrowright_{\mathfrak{s}} \Psi(r) \curvearrowright_{\mathfrak{s}} \rho$.

To find s, repeat the argument, inserting a chain of attacks $a \curvearrowright_f s_{i+1} \curvearrowright_f s_i \curvearrowright_f x$. ⊣ (2·9)

2·10 Corollary *If x is at β, a at α, $a \curvearrowright_f x$, and β is in the abode of α, then x is in the abode of a.*

Proof : for some ρ, $\alpha \curvearrowright_{\mathfrak{s}} \rho \curvearrowright_{\mathfrak{s}} \rho \curvearrowright \beta$; we can therefore find f-recurrent r with $a \curvearrowright_f r \curvearrowright_f x$. ⊣ (2·10)

Many things now fit well together. For example in [3] an operator Γ was introduced: $\Gamma_f(Z) = \{x \mid \exists y(y \in Z \ \& \ y \curvearrowright_f x)\}$.

2·11 Lemma $\Psi[\omega_f(x)] = \omega_{\mathfrak{s}}(\Psi(x))$.

2·12 Lemma $\Psi[\Gamma_f(Z)] = \Gamma_{\mathfrak{s}}(\Psi[Z])$.

2·13 Proposition *For all ν, $\Psi[A^\nu(a, f)] = A^\nu(\Psi(a), \mathfrak{s})$.*

Proof : the previous two lemmata cover the case of 0 and successors. At limits, we must use the analysis of trees. ⊣ (2·13)

2·14 Remark We have made little use of separability: it has been used only to show that any point that vanishes does so at a countable stage, thus giving the upper bound of ω_1 to the score. But for the sake of proving the existence of points of uncountable score, it shouldn't be necessary.

3: Horseshoes

Following a lecture of Jozef Bobok we make a definition. The setting is a compact (Polish) space $\mathcal{A}$ with metric d, a continuous function $f : \mathcal{A} \longrightarrow \mathcal{A}$ and an integer $m \geqslant 2$. We write $\emptyset$ for the sequence of length 0, and $\varnothing$ for the empty set. Of course in many formal presentations of mathematics the two objects are the same.

3·0 Definition An (m, f)-*strong horseshoe* is a sequence $S_0, \dots S_{m-1}$ of pairwise disjoint non-empty closed sets with the property that for all i and j less than m,

$$S_i \subseteq f[S_j].$$

3·1 Remark That is a stronger requirement than the condition met by Smale's original horseshoe, which was that $S_i \cap f[S_j]$ is non-empty for each i and j.

3·2 Lemma *$S_i \cap f^{-1}[S_j] \neq \varnothing$: indeed $f[S_i \cap f^{-1}[S_j]] = S_j$.*

3·3 Lemma *$S_i \cap f^{-1}[S_j] \cap f^{-2}[S_k] \neq \varnothing$; indeed $f^2[S_i \cap f^{-1}[S_j] \cap f^{-2}[S_k]] = S_k$.*

We shall be able to generalise the above, but must first adopt a less cumbersome notation.

3·4 Definition Set $S^\emptyset =_{\rm df} \bigcup_{i<m} S_i$, and for u a sequence of length $k+1$ of numbers less than m, set $S^u =_{\rm df} S^{u\restriction k} \cap f^{-k}[S_{u(k)}]$. For $\alpha \in {}^\omega m$, set $S^\alpha = \bigcap_k S^{\alpha\restriction k}$.

3·5 Proposition *$S^u \neq \varnothing$; indeed $f^k[S^u] = S_{u(k)}$.*

3·6 Proposition *If $u = s^\frown t$, the concatenation of s and t, and $\ell = \ell h(s)$, then $S^u = S^s \cap f^{-\ell}[S^t]$ and $f^\ell[S^u] = S^t$.*

Proof : The first assertion holds by the identity $g^{-1}(C \cap D) = g^{-1}(C) \cap g^{-1}(D)$. For the second, the inclusion from left to right is evident, using the first assertion. Suppose that x is in S^t. Let $v = u \restriction (\ell + 1)$. By Proposition 3·5, there is a y in S^v with $f^\ell(y) = x$; but then $y \in S^s \cap f^{-\ell}(S^t)$, and so, by the first assertion again, $x \in f^\ell[S^u]$. ⊣ (3·6)

3·7 Proposition *Each S^u is a non-empty closed subset of $\mathcal{A}$.*

3·8 Definition $\mathcal{X} =_{\rm df} \{x \in \mathcal{A} \mid \forall k \geqslant 0, f^k(x) \in S^\emptyset\}$.

3·9 Lemma *Each S^α, the intersection along the path α, is non-empty.*

Proof : by compactness. ⊣ (3·9)

3·10 Lemma $\mathcal{X} = \bigcup_\alpha S^\alpha = \bigcap_k \bigcup \{S^u \mid u \in {}^k m\}$.

3·11 Proposition *$\mathcal{X}$ is a closed non-empty subset of $\mathcal{A}$, and is therefore a compact Polish space.*

3·12 Lemma *If $x \in \mathcal{X}$, then $f(x) \in \mathcal{X}$.*

3·13 Lemma *If $x_k \longrightarrow x$ as $k \longrightarrow \infty$, and $x_k \in S^{\alpha\restriction n_k}$, where $n_k \longrightarrow \infty$ with k, then $x \in S^\alpha$.*

Henceforth we work in the space $\mathcal{X}$.

A map to m-Cantor space

Let $\mathcal{C}$ be the product space ${}^{\omega}m$, where m is given the discrete topology. $\mathcal{C}$ is compact by Tychonoff. We shall use Greek letters $\alpha, \beta, \gamma, \rho$ for members of $\mathcal{C}$.

On $\mathcal{C}$ we define the shift function $\mathfrak{s}$ by $\mathfrak{s}(\alpha)(n) = \alpha(n+1)$.

3·14 Definition For $x \in \mathcal{X}$, define $\Psi(x)(n)$ to be the $i < m$ such that $f^n(x)$ is in S_i.

3·15 Proposition *For $x \in \mathcal{X}$, $\Psi(f(x)) = \mathfrak{s}(\Psi(x))$.*

3·16 Remark This means that Ψ is an action map in the sense defined by Akin and Kolyada[1]

3·17 Proposition *If $x \in \bigcap_{k\in\omega} S^{\alpha\restriction k}$, then $\Psi(x) = \alpha$; hence Ψ is surjective.*

[The compactness ensures that the intersection along α is non-empty].

3·18 Proposition *Ψ is continuous.*

We shall say that x *is at* α if $\Psi(x) = \alpha$.

Lifting an attack

3·19 Proposition *$\mathcal{X}$ is $\curvearrowright_f$ closed in the sense that if x is in $\mathcal{X}$ and $x \curvearrowright_f y$ then $y \in \mathcal{X}$.*

3·20 Proposition *If $x \curvearrowright_f y$ then $\Psi(x) \curvearrowright_{\mathfrak{s}} \Psi(y)$.*

Proof : Let $\hat{\varepsilon} =_{\rm df} \min_{i\neq j} d(S_i, S_j)$; thus $\hat{\varepsilon} > 0$ and has the property that two points in $S^{\emptyset}$ within distance $\hat{\varepsilon}$ of each other are in the same S_i.

Fix a natural number k. We seek ℓ such that $f^{\ell}(x)$ shadows y for k steps, in the sense that for $n < k$, $\Psi(f^{\ell}(x))(n) = \Psi(y)(n)$.

By the continuity of f, f^2 … and f^{k-1} at y, there are $\delta_i > 0$ such that for each $i < k$

$$d(a, y) < \delta_i \implies d(f^i(a), f^i(y)) < \hat{\varepsilon}$$

Take δ to be the minimum of the δ_i's. Pick ℓ exceeding k and large enough so that $d(f^{\ell}(x), y) < \delta$. ⊣ (3·20)

3·21 Corollary *If r is f-recurrent then $\Psi(r)$ is $\mathfrak{s}$-recurrent.*

Proof : Let $\Psi(r) = \rho$. $r \curvearrowright_f r$, so $\rho \curvearrowright_{\mathfrak{s}} \rho$. ⊣ (3·21)

3·22 Proposition *Suppose that $\alpha \curvearrowright_{\mathfrak{s}} \beta$, and that x is at α. Then we can find y at β such that $x \curvearrowright_f y$.*

Proof : Let $\beta \restriction k = s^{n_k}(\alpha) \restriction k$.Then put $x_k = f^{n_k}(x)$. Each x_k is in $S^{\beta\restriction k}$; by compactness some subsequence of the x's converges, to y say. Then $y \in S_\beta$ and $x \curvearrowright_f y$. ⊣ (3·22)

3·23 Corollary *If x is at α and the abode of α is empty, so is the abode of x.*

Proof : the abode of a point is empty iff it attacks no recurrent points. ⊣ (3·23)

The clauses (2·0·0), (2·0·1) and (2·0·2) are established in the present setting by 3·15, 3·20 and 3·22.

4: The effect of equicontinuity

Suppose now that at every point x of $\mathcal{X}$, f is equicontinuous in the sense that

$$\forall\varepsilon > 0\, \exists\delta > 0\, \forall n\, \forall y\, \big[d(x,y) < \delta \implies d(f^n(x), f^n(y)) < \varepsilon\big].$$

We derive clause (2·0·3):

4·0 Lemma *Suppose that $\alpha \curvearrowright_{\mathfrak{s}} \beta \curvearrowright_{\mathfrak{s}} \gamma$ and that a is at α, c at γ and $a \curvearrowright_f c$. Then there is a point b at β with $a \curvearrowright_f b \curvearrowright_f c$.*

Proof : Given k, a positive integer, there are (large) integers n_k and m_k such that for each $i < k$,

$$\gamma(i) = \beta(n_k + i) = \alpha(m_k + n_k + i)$$

and such that $f^{m_k+n_k}(a) \longrightarrow c$.

Set $b_k = f^{m_k}(a)$. Some subsequence, say for $k \in B \in [\omega]^\omega$, of those converges, to b, say. We know that $f^{n_k}(b_k) \longrightarrow d$. We shall use the equicontinuity of f at b to show that $f^{n_k}(b) \longrightarrow d$.

Given ε, we seek K such that for $k > K$, $k \in B$, $d(f^{n_k}(b), d) < \varepsilon$. We know that there is a K_0 such that for all $k \geqslant K_0$, $d(f^{n_k}(b_k), c) < \varepsilon/2$. Using the equicontinuity at b, we know that there is a δ such that if $d(z, b) < \delta$, then for all k, $d(f^{n_k}(z), f^{n_k}(b)) < \varepsilon/2$. For large enough $k \in B$, we shall indeed have $d(b_k, b) < \delta$. ⊣ (4·0)

4·1 Remark Equicontinuity seems formally too strong, as for given k we are only interested in the point $z = b_k$.

Finally we mention a variant of (2·0·3), of which the proof has an interesting feature.

4·2 Lemma *Suppose that $\alpha \curvearrowright_{\mathfrak{s}} \beta$ and that y is at β. Then there is an x at α with $x \curvearrowright_f y$.*

Proof : By Proposition 3·6, for given k and n_k with $\mathfrak{s}^{n_k}(\alpha) \restriction k = \beta \restriction k$, we may find $x_k \in S^{\alpha \restriction n_k}$ with $f^{n_k}(x_k) = y$ (and not just near y !). Let x be the limit of some convergent subsequence of the x_k's. x will be at α by Lemma 3·13. Then by the equicontinuity of f at x, given $\varepsilon > 0$ there is a $\delta > 0$ such that for each k and z $d(x, z) < \delta \implies d(f^{n_k}(x), f^{n_k}(z)) < \varepsilon$; taking z to be an x_k in the subsequence that is suitably near x, we see that $d(f^{n_k}(x), y) < \varepsilon$, and so $x \curvearrowright_f y$, as required. ⊣ (4·2)

REFERENCES

[1] E. Akin and S. Kolyada, Li-Yorke sensitivity, *Nonlinearity* **16** (2003) 1421–1433.

[2] A. R. D. Mathias, Recurrent points and hyperarithmetic sets, in *Set Theory, Techniques and Applications, Curaçao 1995 and Barcelona 1996 conferences,* edited by C. A. Di Prisco, Jean A. Larson, Joan Bagaria and A. R. D. Mathias, Kluwer Academic Publishers, Dordrecht, Boston, London, 1998, 157–174; MR 2000d:03097.

[3] A. R. D. Mathias, Delays, recurrence and ordinals, *Proc. London Math. Soc. (3)* **82** (2001) 257–298; MR 2001j:03087.

[4] A. R. D. Mathias, Analytic sets under attack, *Math. Proc Cam. Phil. Soc*, to appear.