Rudimentary recursion and provident sets

A. R. D. Mathias
ERMIT, Université de la Réunion

Abstract. We introduce the class of rudimentarily recursive functions, and study those sets, which we call provident, which are non-empty, transitive and closed under all rudimentarily recursive functions, allowing parameters from within the set in question. We identify a single rudimentary recursion, with parameter, to instances of which all others reduce; we obtain various characterizations of provident sets, showing in particular that the segment J_ν of the Jensen hierarchy is provident if and only if $\omega\nu$ is an indecomposable ordinal; and we find uniform affine bounds on the rate of growth of rudimentarily recursive functions.

In the closing sections we look briefly at various more liberal recursions, we give models of Zermelo set theory that fail in various ways to support rudimentary recursions, and we offer improvements on existing definitions of truth predicates.

Contents

A sequel, *Provident sets and rudimentary set forcing*, will show that forcing can be done over any transitive set or class M which is provident, with a poset $\mathbb{P}$ that is a member of M; that the Forcing Theorem will hold for $\dot{\Delta}_0$ sentences of the forcing language; and that the generic extension will be provident.

0: Introduction

The Σ_1 recursion theorem of Kripke-Platek set theory KP proves for G a Σ_1 function that if G is total, so is the function F given by the recursion

$$F(x) = G(F \restriction x),$$

and further F is provably equal to a Σ_1 function.

We wish to consider this theorem in the case that the defining function G is not merely Σ_1 but actually rudimentary in the sense of Jensen [J2]. In such cases we shall speak of F as given by a *rudimentary recursion*, or, more briefly, that F *is rud rec*. We shall also use this terminology when F is intended to be a function defined on On rather than on V, or defined by recursion on other well-founded relations related to the epsilon relation.

Some rudimentary recursions

0·0 EXAMPLE The definition of *rank*:

$$\varrho(x) = \bigcup\{\varrho(y) + 1 \mid_y y \in x\}$$

0·1 EXAMPLE The definition of *transitive closure*:

$$\mathrm{tcl}(x) = x \cup \bigcup\{\mathrm{tcl}(y) \mid_y y \in x\}$$

0·2 EXAMPLE Let $\mathcal{S}(x)$ be the set of finite subsets of x. *Restricted to ordinals*, this has a rudimentarily recursive definition:

$$\mathcal{S}(0) = \{\varnothing\}; \quad \mathcal{S}(\zeta + 1) = \mathcal{S}(\zeta) \cup \{x \cup \{\zeta\} \mid_x x \in \mathcal{S}(\zeta)\}; \quad \mathcal{S}(\lambda) = \bigcup_{\nu<\lambda} \mathcal{S}(\nu).$$

0·3 EXAMPLE Jensen in [J2] gives a single rudimentary function that can be used to generate the constructible universe. We recall below the definition of a unary rudimentary function $\mathbb{T}$, introduced in *Weak Systems* [M3], such that the following rudimentary recursion on On, the class of von Neumann ordinals,

$$T_0 = \varnothing; \qquad T_{\nu+1} = \mathbb{T}(T_\nu); \qquad T_\lambda = \bigcup_{\nu<\lambda} T_\nu$$

which can be said in one breath as

$$T_\zeta = \bigcup_{\nu<\zeta} \mathbb{T}(T_\nu)$$

generates L and the Jensen hierarchy in that $L = \bigcup_{\nu \in ON} T_\nu$, and $J_\nu = T_{\omega\nu}$.

0·4 REMARK If we recursively define $T(x) = \bigcup_{y\in x} \mathbb{T}(T(y))$, then $T(x)$ always equals $T_{\varrho(x)}$. More generally, if G is a (class) (rudimentary) function such that $u \subseteq G(u)$, and we define $E_0 = \varnothing$, $E_{\nu+1} = G(E_\nu)$, $E_\lambda = \bigcup_{\nu<\lambda} E_\nu$ by recursion on the ordinals, and $F(x) = \bigcup_{y\in x} G(F(y))$ by set recursion, then for all x, $F(x) = E_{\varrho(x)}$.

0·5 EXAMPLE To form $L(d)$, the constructible closure of d, a transitive set, requires a rud recursion in the parameter d: define

$$D(x) = d \cup \bigcup_{y\in x} \mathbb{T}(D(y)).$$

Then $D(x) = D_{\varrho(x)}$ where $D_0 = d$; $D_{\nu+1} = \mathbb{T}(D_\nu)$; $D_\lambda = \bigcup_{\nu<\lambda} D_\nu$, which is the usual ordinal recursion for this purpose. $L(d) = \bigcup_x D(x) = \bigcup_\nu D_\nu$.

In fact, for the purposes of the present paper a different recursion proves desirable, and will be introduced in due course.

0·6 EXAMPLE Suppose we are making a forcing extension using a notion of forcing $\mathbb{P}$ that is a set of the ground model, assumed transitive. In the theory of forcing, a member y of the ground model is represented by the term $\hat{y}$ of the language of forcing, given by the recursion

$$\hat{y} =_{\mathrm{df}} \{(\mathbb{1}^{\mathbb{P}}, \hat{x}) \mid_x x \in y\}$$

This is a rudimentary recursion in a parameter, being of the form

$$F(a) = G(\mathbb{1}^{\mathbb{P}}, F \restriction a)$$

where G is the rudimentary function $(\mathbb{1}^{\mathbb{P}}, a) \mapsto \{\mathbb{1}^{\mathbb{P}}\} \times \mathrm{Im}(a)$: though it would be a simple matter to specify that $\mathbb{1}^{\mathbb{P}}$ is always to be some hereditarily finite set, for example 1, when G could be rewritten as a pure rud function.

0·7 EXAMPLE If M is an (intransitive) elementary submodel of a transitive set or class, then the Mostowski collapsing isomorphism ϖ_M is given by the recursion

$$\varpi_M(x) = \{\varpi_M(y) \mid_y y \in x \cap M\}$$

so that, in some sense, ϖ_M is rudimentarily recursive in the predicate M.

0·8 EXAMPLE If $\mathcal{G}$ is a generic filter on a notion of forcing $\mathbb{P}$ in a transitive model M, and we follow Shoenfield in treating all members of M as $\mathbb{P}$-names, the function $\mathrm{val}_{\mathcal{G}}(\cdot)$ defined for $a \in M$ is given by a rudimentary recursion with $\mathcal{G}$ as a parameter.

$$\mathrm{val}_{\mathcal{G}}(b) =_{\mathrm{df}} \{\mathrm{val}_{\mathcal{G}}(a) \mid_a \exists p{:}{\in}\mathcal{G}\ (p, a) \in b\}$$

The generic extension $M[\mathcal{G}]$ is then be defined as $\{\mathrm{val}_{\mathcal{G}}(a) \mid_a a \in M\}$.

0·9 REMARK Note that the definition of the forcing relation $\Vdash$ has not been invoked in making these definitions, but its properties would be needed to show that $M[\mathcal{G}]$ has interesting properties.

0·10 REMARK Close scrutiny reveals that the function $\mathrm{val}_{\mathcal{G}}(\cdot)$ combines two functions, which we might call transforming and collapsing, and when considering forcing in certain contexts, to be explored in a companion paper *Provident sets and rudimentary set forcing*, [M4], there are grounds for separating the two functions. For example, if $\mathcal{G}$ is $(M, \mathbb{P})$-generic, one might first define for $x \in M$

$$\tilde{\pi}(x) = \{(\mathbb{1}^{\mathbb{P}}, \tilde{\pi}(a)) \mid_{p,a} (p, a) \in x \ \&\ p \in \mathcal{G}\},$$

thus transforming $\mathbb{P}$-names to $\mathbb{2}$-names;‡ and then one would collapse the class of pure $\mathbb{2}$-names, to obtain the desired generic extension, by setting for $x \in \mathrm{Im}\,(\tilde{\pi})$,

$$\varpi(x) = \{\varpi(y) \mid_y (\mathbb{1}^{\mathbb{P}}, y) \in x\},$$

which of course is the inverse of the function $x \mapsto \hat{x}$ when the latter is taken to be defined on $M[\mathcal{G}]$. Both recursions are rudimentary in appropriate parameters or classes.

0·11 EXAMPLE The relation $x \in^\star y$, meaning x is in the transitive closure of y, is given by a rud recursion on the second variable y, the first variable x remaining free:

$$x \in^\star y \iff x \in y \lor \exists z{:}{\in}y\ x \in^\star z$$

0·12 EXAMPLE Ordinal addition is given by the recursion

$$A(\alpha, 0) = \alpha;\ A(\alpha, \beta + 1) = A(\alpha, \beta) + 1;\ A(\alpha, \lambda) = \bigcup_{\nu<\lambda} A(\alpha, \nu)$$

That is again a rud recursion on the second variable, the first remaining free.

‡ Historical influences are slightly confusing our notation here, as $\mathbb{2}$ is the complete Boolean algebra naturally associated to the partial order whose sole member is $\mathbb{1}$.

Rud rec functions which are not rudimentary

0·13 EXAMPLE In *Weak Systems* [M3], §12, a transitive model of ZC is given in which TCo fails. Thus tcl though rud rec, cannot be rud. Note that the rank of $\mathrm{tcl}(x)$ always equals that of x.

0·14 EXAMPLE Consider for any limit ordinal λ, "Model $\mathbf{M}_{13,\lambda}$" studied in §7 of *Weak Systems*, which is supertransitive and a proper class but which contains only the ordinals $< \lambda$, and is defined as the union of all transitive sets u whose intersection with λ is bounded strictly below λ. Every transitive model of Zermelo set theory (even without the axiom of infinity) is rudimentarily closed, essentially because GJ_0 is a subsystem of Z; so this model, in which TCo holds, shows that ZC is incapable of proving that rank is total and that rank is a rud rec function that is not rudimentary.

0·15 EXAMPLE The function $\zeta \mapsto 2{\cdot}\zeta$ is given by a rudimentary recursion. Nevertheless it is not rudimentary, for the rud closure of $\{\omega\}$ has ω as a member but, by Gandy [G], not $EVEN =_{\mathrm{df}} \{2 \cdot n \mid_n n \in \omega\}$.

0·16 EXAMPLE The characteristic function of *EVEN* is given by a rud recursion on ω:

$$\chi(0) = 1; \qquad \chi(n+1) = 1 \smallsetminus \chi(n).$$

Note that $\chi \restriction \omega \notin \mathrm{rud\ cl} J_1 \cup \{\omega\}$.

0·17 REMARK Corollary 14·5 of *Weak Systems* shows that J_2 is not the rud closure of $J_1 \cup \{\omega\}$, J_1 not being a member of that latter set; but J_2 is the rud rec closure of $J_1 \cup \{\omega\}$; indeed of $\omega + 1$.

0·18 REMARK J_2 has recently been the object of study by Nik Weaver: *Analysis in J_2*.

Some functions which are not rud rec

0·19 REMARK The function $\beta \mapsto \beta + \omega$, simple though it be, is *not* given by a pure rudimentary recursion; still less are the other functions of ordinal arithmetic; nor is the J hierarchy. The reason is that any rud function G only raises rank by a finite amount k, a uniform bound for all arguments; which we may call the *rudimentary constant* of G; from that it will follow that for a pure rud rec function F, for each argument x, $\varrho(F(x)) < \varrho(x) + \omega$.

0·20 REMARK In a notation that we develop in §5, if F is $\varnothing$-rud-rec, $F \restriction P^c_\lambda \subseteq P^c_\lambda$, so for x of limit rank, $\varrho(F \restriction x) = \varrho(x)$. When a parameter p is involved, the same equality will hold for x of limit rank at least the ordinal product $\varrho(p) \cdot \omega$.

0·21 REMARK The function g given by the recursion

$$g(0) = 1; \qquad g(\nu + 1) = f(g(\nu)); \qquad g(\lambda) = \sup g``\lambda$$

where $f(\xi) = 2 \cdot \xi$ is given by a rud-rec recursion, but not by a rud recursion, as its rate of growth for finite arguments is too great.

0·22 REMARK We cannot usually expect that a Σ_1 total function is total when relativised to a rud closed set; but in the case of rudimentary recursions and sets in one of the standard hierarchies, this will be true provided we "get off to a good start," by which we mean the phenomenon described in Proposition 5·44.

0·23 REMARK The function $x \mapsto \mathcal{S}(x)$ is not given by a pure rud rec function, as we shall see below by estimating the rate of growth of its cardinality for $x \in \mathbf{HF}$. But we could define it by a recursion with parameter ω by remarking that for k a positive integer,

$$[a]^{k+1} = \Big\{x \cup \{y\} \,\Big|_{x,y}\, x \in [a]^k \;\&\; y \in \bigcup [a]^k\Big\} \smallsetminus [a]^k.$$

0·24 REMARK The functions $x \mapsto \mathcal{S}(x)$ and $\beta \mapsto \beta + \omega$ are examples of recursions of Type III, to be defined in §4. Proposition 5·51 will show that provident sets are closed under recursions of this type.

1: Review of the elementary theory of rudimentary functions

We regard set theory as formalised in a syntax with a class-forming operator and both restricted and unrestricted quantifiers: Appendix One gives details to supplement those sketched in the present section.

All our systems of set theory will have among their axioms those of classical propositional and predicate logic, the axioms relating restricted quantifiers to unrestricted ones,

$$\forall x :\in y \ \Phi \iff \forall x(x \in y \implies \Phi)$$
$$\exists x :\in y \ \Phi \iff \exists x(x \in y \ \& \ \Phi)$$
$$x \in \{y \mid \Phi\} \iff [\Phi\, y/x]$$

where $[\Phi\, y/x]$ denotes the result of substituting the variable x for the free occurrences of the variable y in the formula Φ, bound occurrences of y in Φ being first changed to an as yet unused variable; and will have among their set-theoretical axioms

Extensionality:	$(\forall w :\in x \ \ w \in y \,\&\, \forall w :\in y \ \ w \in x) \implies x = y$
Empty Set:	$\varnothing \in V$
Pair:	$\{x, y\} \in V$
Difference:	$x \setminus y \in V$
Union:	$\bigcup x \in V$

Call the system so far S_0.

Suitable terms

1·0 Definition A wff or class is Δ_0 if no unrestricted quantifiers occur in it. If S is any system of set theory containing S_0 we say that a class A or a wff Φ is Δ_0^{S} iff there is a class B or a wff Ψ such that $\vdash_{\mathsf{S}} A = B$ or $\vdash_{\mathsf{S}} \Phi \iff \Psi$ respectively.

1·1 Definition A class A is S-*suitable* if $\vdash_{\mathsf{S}} A \in V$ and for each Δ_0 wff Ψ and variable w **not occurring freely in** A, $\forall w :\in A \ \Psi$ is Δ_0^{S}.

This notion is important in building a calculus of Δ_0 wffs, which we now do.

1·2 Proposition *If Φ and Ψ are Δ_0^{S}, so are $\exists w :\in z \ \Phi$, $\forall w :\in z \ \Phi$, where w and z are distinct variables, $(\Phi \,\&\, \Psi)$, $\neg\Phi$ and $x \in \{y \mid \Phi\}$.*

1·3 Definition Let x be a variable, B a class and Φ a wff. Then $[\Phi\, x/B]$ is the result of

i) changing all bound occurrences variables in Φ to occurrences of variables not occurring in B or free in Φ;
ii) replacing all free occurrences of x in the new formula by B;
iii) expanding occurrences of the subformulæ "$B \in t$", "$B = t$", "$t = B$", "$\forall y :\in B$ " according to the definitions above.

Similarly one may define $[A\, x/B]$ for A a class. Expressions such as $[\Phi A/B]$ are not defined.

1·4 Proposition *Let A be S-suitable.*
(i) if Φ is Δ_0^{S}, so is $\exists w :\in A \ \Phi$, provided w is not free in A;
(ii) $w \in A$, $w = A$, $A \in w$ are Δ_0^{S}, even if w occurs in A;
(iii) if Φ is Δ_0, $[\Phi\, x/A]$ is Δ_0^{S};
(iv) if Φ is Δ_0^{S}, so is $[\Phi\, x/A]$.

It is necessary to prove (iii) before (iv), since a subformula of a Δ_0^{S} formula need not be Δ_0^{S}.

1·5 Proposition *If A and B are S-suitable, so is $[B\, x/A]$.*

The system S_0 is too weak to support the study of the class of rudimentary functions that we are about to describe; accordingly we specify two progressively stronger axiomatic systems, DB_0 and then GJ_0.

The systems DB_0 and GJ_0

1·6 DEFINITION The system DB_0 has as its set-theoretic axioms Extensionality and these nine:

$$\begin{array}{lll} \varnothing \in V & \bigcup x \in V & x \cap \{(a,b)_2 \mid_{a,b} a \in b\} \in V \\ \{x,\, y\} \in V & \mathrm{Dom}\,(x) \in V & \{(b,a,c)_3 \mid_{x,y,z} (a,b,c)_3 \in y\} \in V \\ x \smallsetminus y \in V & x \times y \in V & \{(b,c,a)_3 \mid_{x,y,z} (a,b,c)_3 \in z\} \in V \end{array}$$

The following result, of which many variants appear in the literature, presumably goes back to Bernays.

1·7 THEOREM *All instances of Δ_0 separation are provable in the system DB_0*

1·8 DEFINITION We shall call a function of the form $x \mapsto x \cap A$, where A is a class, a *separator*, or a Δ_0-*separator* if A is a Δ_0 class.

1·9 DEFINITION The axioms of the system GJ_0 are those of DB_0 plus the single axiom called R_8 in [M3]:

$$(R_8) \qquad \{x``\{w\} \mid w \in y\} \in V$$

A set of generators for the class of rudimentary functions

We introduce the rudimentary functions $R_0, \dots R_8$ and certain auxiliary functions $A_0 \dots A_{14}$ generated by them under composition: this is not the shortest possible list, but one that conveniently extends the list, given in Definition 1·6, that generates the Δ_0 separators.

$R_0(x,y) = \{x,y\}$

$A_0(x) = \{x\}\,[= R_0(x,x)]$

$A_1(x,y) = (x,y)_2\,[= R_0(A_0(x), R_0(x,y))]$

$A_2(x,y,z) = \{x, (y,z)_2\}$

$A_3(x,y,z) = (x,y,z)_3\,[= A_1(x, A_1(y,z))]$

$R_1(x,y) = x \setminus y$

$A_4(x,y) = x \cap y\,[= x \setminus (x \setminus y)]$

$A_5(x) = \varnothing\,[= x \setminus x]$

$A_6(x) = x\,[= x \setminus \varnothing]$

$R_2(x) = \bigcup x$

$R_3(x) = \mathrm{Dom}\,(x)$

$R_4(x,y) = x \times y$

$R_5(x) = x \cap \{(a,b)_2 \mid_{a,b} a \in b\}$

$A_7(x) = eps \restriction x\,[= R_5(\bigcup x \times x)]$

$R_6(x) = \{(b,a,c)_3 \mid_{a,b,c} (a,b,c)_3 \in x\}$

$R_7(x) = \{(b,c,a)_3 \mid_{a,b,c} (a,b,c)_3 \in x\}$

$A_8(x) = \{(a,c,b)_3 \mid_{a,b,c} (a,b,c)_3 \in x\}\,[= R_6(R_7(R_7(x)))]$

$A_9(x) = x^{-1}\,[= \mathrm{Dom}\,(\{(a,c,b)_3 \mid_{a,b,c} (a,b,c)_3 \in \{\varnothing\} \times x\}) = R_3(A_8(R_4(A_0(A_5(x)), x)))]$

$A_{10}(x) = \mathrm{Im}\,(x)\,[= \mathrm{Dom}\,(x^{-1})]$

$A_{11}(x,y) = eps \cap (x \times y)\,[= R_5(x \times y)]$

$A_{12}(x,y) = \{w \mid_w x \in w \in y\}\,[= \mathrm{Dom}\,(A_{11}(\{x\} \times y))]$

$A_{13}(x,y) = id \cap (x \times y)$

$A_{14}(x,y) = x``\{y\}\,[= \mathrm{Dom}\,((x \cap ([\bigcup\bigcup x] \times \{y\}))^{-1})]$

$R_8(x,y) = \{x``\{w\} \mid_w w \in y\}$

Separators and basic functions

1·10 DEFINITION Let $\mathcal{B}$, the class of *basic functions*, be the closure of $R_0 \dots R_7$ under composition.

1·11 PROPOSITION *For each $\dot{\Delta}_0$ class A the map $x \mapsto x \cap A$ is in $\mathcal{B}$.*

A proof will be found in Appendix Two.

Companions for rudimentary functions

1·12 DEFINITION Let $\mathcal{R}$ be the closure of $R_0 \ldots R_8$ under composition. $\mathcal{R}$ is the class of *rudimentary functions.*

The collection of functions in $\mathcal{R}$ is closed under formation of images: by which is meant that if F is in $\mathcal{R}$ so is $x \mapsto F``x$. To prove this we introduce the notion of a *companion*—we will actually have two such notions—and establish the Gandy–Jensen Lemma.

Let S be some system of set theory extending DB_0, and let G and F be Δ_0 classes such that S proves that both G and F are total functions.

DEFINITION G is a *1-companion* of F in S if G is S-suitable and

$$\vdash_{\mathsf{S}} \vec{x} \in \vec{u} \implies F(\vec{x}) \downarrow\in G(\vec{u})$$

DEFINITION H is a *2-companion* of F in S if H is S-suitable and

$$\vdash_{\mathsf{S}} \vec{x} \in \vec{u} \implies F(\vec{x}) \downarrow\subseteq H(\vec{u})$$

where $\vec{x} \in \vec{u}$ abbreviates $x_1 \in u_1 \;\&\; \ldots x_n \in u_n$ for an appropriate n.

1·13 PROPOSITION *If G^1 is a 1-companion of G in S and H^1 is a 1-companion of H in S, then $G^1 \circ H^1$ is a 1-companion of $G \circ H$ in S.*

The function $F``$, if available in S, is the best 1-companion of F in S, and in favourable cases separators may be used to reduce a given 1-companion F^1 of F to that one, since

$$\vdash_{\mathsf{S}} F``a = F^1(a) \cap \{y \mid \exists x :\in a \;\, y = F(x)\}$$

so that if F is given by an S-suitable term,

$$\vdash_{\mathsf{S}} y = F(x) \iff \forall w :\in y \;\, w \in F(x) \;\&\; \forall w :\in F(x) \;\, w \in y$$

1·14 PROPOSITION *Each of the functions $R_0, \ldots, R_7$ and A_{14} has a 2-companion in DB.*

Proof :

R_0: $a \in x \,\&\, b \in y \implies \{a, b\} \subseteq x \cup y = \bigcup\{x, y\}$.
R_1: $a \in x \,\&\, b \in y \implies a \setminus b \subseteq a \subseteq \bigcup x$.
R_2: $a \in x \implies \bigcup a \subseteq \bigcup\bigcup x$.
R_3: $a \in x \implies \mathrm{Dom}\,(a) \subseteq \bigcup\bigcup x$.
R_4: $a \in x \,\&\, b \in y \implies a \times b \subseteq \bigcup x \times \bigcup y$.
R_5: $t \in x \implies t \cap \{(a, b)_2 \mid a \in b\} \subseteq t \subseteq \bigcup x$.
R_6: $t \in x \implies \{(b, a, c)_3 \mid (a, b, c)_3 \in t\} \subseteq \mathrm{Im}\,(\mathrm{Dom}\,(\bigcup x)) \times (\mathrm{Im}\,(\bigcup x) \times \mathrm{Dom}\,(\mathrm{Dom}\,(\bigcup x)))$,
[by reasoning similar to that given below for R_7.]
R_7: $t \in x \implies \{(b, c, a)_3 \mid (a, b, c)_3 \in t\} \subseteq \mathrm{Im}\,(\mathrm{Dom}\,(\bigcup x)) \times (\mathrm{Dom}\,(\mathrm{Dom}\,(\bigcup x)) \times \mathrm{Im}\,(\bigcup x))$.

To see this, note that $\{(b, c, a)_3 \mid (a, b, c)_3 \in t\} \subseteq \mathrm{Im}\,(\mathrm{Dom}\,(t)) \times (\mathrm{Dom}\,(\mathrm{Dom}\,(t)) \times \mathrm{Im}\,(t))$, and apply these principles: $t \in x \implies t \subseteq \bigcup x$; $t \subseteq s \implies \mathrm{Dom}\,(t) \subseteq \mathrm{Dom}\,(s)$; $t \subseteq s \implies \mathrm{Im}\,(t) \subseteq \mathrm{Im}\,(s)$; and $t \subseteq s \,\&\, v \subseteq u \implies t \times v \subseteq s \times u$.

A_{14}: $a \in x \;\&\; b \in y \implies a``\{b\} \subseteq \mathrm{Im}\,(\bigcup x)$. $\dashv$

1·15 REMARK The above 2-companions are generated by four functions, namely, Im , Dom , $\bigcup$ and $\times$. We can get that down to two, $\bigcup$ and $\times$, by using the above principles. For u transitive, a single generator, the function $u \mapsto u^\star \;=_{\mathrm{df}}\; u \cup [u]^{\leqslant 2} \cup (u \times u)$ is enough.

1·16 PROPOSITION *If F has a 1-companion F^1 then $\bigcup F^1$ is a 2-companion of F.*

1·17 PROPOSITION *If G has a 2-companion G^2 and H has a 1-companion H^1, then $G^2 \circ H^1$ is a 2-companion of $G \circ H$.*

The Gandy–Jensen Lemma

The Gandy–Jensen Lemma is the core of the proof that $\mathcal{R}$ is closed under formation of images. Versions of it are to be found in the papers of Gandy [G] and Jensen [J2]. We discuss it only for 1-ary functions. The extension to n-ary functions poses no problems.

1·18 THE GANDY–JENSEN LEMMA *Let* S *be a system extending* DB_0. *Suppose that* H *is a 2-companion of* F *in* S, *and that '*$a \in F(b)$*' is* Δ_0^{S}. *Then* F *is generated by composition from* H *and members of* $\mathcal{B}$, *and so is* S*-suitable; if in addition* S *extends* GJ, *then* $\vdash_{\mathsf{S}} F``x \in V$ *and* $F``$ *(as a function) is generated by* H *and members of* $\mathcal{R}$ *and (as a term) is* S*-suitable and is a 1-companion of* F *in* S.

Proof : We have

$$\vdash_{\mathsf{S}} x \in u \implies F(x) \subseteq H(u).$$

Working in S, form

$$h(u) =_{\mathrm{df}} \big(H(u) \times u\big) \cap \{(a,b)_2 \mid_{a,b} b \in u \ \&\ a \in F(b)\}.$$

Since "$a \in F(b)$" is Δ_0^{S} and for each Δ_0 A, the function $x \mapsto x \cap A$ is in $\mathcal{B}$ and is DB-suitable, we have that h is S-suitable, and is generated by H and functions in $\mathcal{B}$.

Now note that for $b \in u$, $F(b) = h(u)``\{b\} = A_{14}(h(u), b)$, so $F$ is built from $H$ and functions in $\mathcal{B}$; if $R_8$ is available in the system $\mathsf{S}$, we may argue further that $F``u = R_8(h(u), u)$ so $F``$ is built from $H$ and rudimentary functions, and is thus $\mathsf{S}$-suitable; hence $\vdash_{\mathsf{S}} F``u \in V$, and this function $F``$ now forms a 1-companion of F in S. ⊣

1·19 PROPOSITION *R_8 has a 2-companion in* GJ.

Proof : By the Gandy–Jensen lemma, $A_{14}``$ is GJ-suitable, and so

$$a \in x \ \&\ b \in y \implies R_8(a,b) = \{A_{14}(a,w) \mid w \in b\} = A_{14}``(\{a\} \times b) \subseteq A_{14}``(x \times \bigcup y). \qquad ⊣\ (1{\cdot}19)$$

1·20 COROLLARY *R_8 has a 1-companion in* GJ.

Proof : by the Gandy–Jensen Lemma. ⊣ (1·20)

1·21 THEOREM *$\mathcal{R}$ is closed under formation of images and of unions of images.*

Proof : We have seen that each of $R_0, \dots R_8$ has a 1-companion in GJ; the class of functions possessing a 1-companion is closed under composition, and hence each function in $\mathcal{R}$ has a 1-companion in GJ; but if G is a 1-companion of F then $u \mapsto \bigcup(G(u))$ is a 2-companion of F. Hence each function F in $\mathcal{R}$ has a 2-companion in GJ; each such function is GJ-suitable, Proposition 1·5 proving the survival of suitability under composition, and so by the Gandy–Jensen lemma, $F``$ is in $\mathcal{R}$; composition with $\bigcup$ yields the last clause. ⊣

1·22 REMARK Gandy shows in [G] that these three are equivalent: (i) F is rudimentary; (ii) "$a \in F(b)$" is Δ_0 and F has a 1-companion in GJ; (iii) "$a \in F(b)$" is Δ_0 and F has a 2-companion in GJ.

1·23 REMARK Gandy in [G] and Jensen in [J2] supply other characterisations of $\mathcal{R}$ and other axiomatisations of GJ.

Rudimentary recursion on the ordinals

1·24 REMARK For $G(p,f)$ a binary rudimentary function, define $G'(p,f) = G(p,f) \cap \{z \mid \mathrm{Dom}\, f \in On\}$. Then G' is rudimentary, by Proposition 1·11; and if for a set p we recursively define $F(x) = G'(p, F \restriction x)$, then F is p-rudimentarily recursive and

$$F(x) = \begin{cases} G(p, F \restriction x) & \text{if } x \in On \\ \varnothing & \text{otherwise.} \end{cases}$$

2: A single generating function for $\mathbf{rud}(u)$

In developing further properties of the class of rudimentary functions we shall use the function $\mathbb{T}$ introduced in Definition 2·73 of *Weak Systems*.

The function $\mathbb{T}$

2·0 DEFINITION

$$
\begin{aligned}
\mathbb{T}(u) =_{\mathrm{df}}\ & u \cup \{u\} \\
& \cup [u]^1 \cup [u]^2 \\
& \cup \{x \smallsetminus y \mid_{x,y} x, y \in u\} \\
& \cup \{\textstyle\bigcup x \mid_x x \in u\} \\
& \cup \{\mathrm{Dom}\,(x) \mid_x x \in u\} \\
& \cup \{u \cap (x \times y) \mid_{x,y} x, y \in u\} \\
& \cup \{x \cap \{(a,b)_2 \mid_{a,b} a \in b\} \mid_x x \in u\} \\
& \cup \{u \cap \{(b,a,c)_3 \mid_{a,b,c} (a,b,c)_3 \in x\} \mid_x x \in u\} \\
& \cup \{u \cap \{(b,c,a)_3 \mid_{a,b,c} (a,b,c)_3 \in x\} \mid_x x \in u\} \\
& \cup \{x``\{w\} \mid_{x,w} x \in u, w \in u\} \\
& \cup \Big\{u \cap \{x``\{w\} \mid_w w \in y\} \Big|_{x,y} x, y \in u\Big\}.
\end{aligned}
$$

2·1 REMARK The successive lines of the definition of $\mathbb{T}$, after the first, may be written more prosaically as $R_0``(u \times u)$, $R_1``(u \times u)$, $R_2``u$, $R_3``u$, $\{u \cap R_4(x,y) \mid_{x,y} x, y \in u\}$, $R_5``u$, $\{u \cap R_6(x) \mid_x x \in u\}$, $\{u \cap R_7(x) \mid_x x \in u\}$, $A_{14}``(u \times u)$ and $\{u \cap R_8(x,y) \mid_{x,y} x, y \in u\}$. It will be notationally convenient to treat all these functions as having three variables, so let us define $S_i(u; x, y) := R_i(x,y)$ for $i = 0, 1$; $S_i(u; x, y) := R_i(x)$ for $i = 2, 3, 5$; $S_i(u; x, y) := u \cap R_i(x,y)$ for $i = 4, 8$; $S_i(u; x, y) := u \cap R_i(x)$ for $i = 6, 7$; and $S_9(u; x, y) := A_{14}(x,y)$. Then each of those lines now takes the form $S_i``(\{u\} \times (u \times u))$ for some i.

We have proved the first clause of the following, and the others are easy.

2·2 PROPOSITION *$\mathbb{T}$ is rudimentary, $u \subseteq \mathbb{T}(u)$ and $u \in \mathbb{T}(u)$. Further, if u is transitive, then $\mathbb{T}(u)$ is a set of subsets of u, and hence $\mathbb{T}(u)$ is transitive.*

2·3 REMARK It will not in general be true that $u \subseteq v \implies \mathbb{T}(u) \subseteq \mathbb{T}(v)$, the problem being that $u \in \mathbb{T}(u)$, but if v is countably infinite, so is $\mathbb{T}(v)$ which therefore cannot contain all the subsets of v. Fortunately, $u \subseteq \mathbb{T}(u) \subseteq \mathbb{T}^2(u) \ldots$

2·4 LEMMA *If x and y are in u, then $R_0(x,y)$, $R_1(x,y)$, $R_2(x)$, $R_3(x)$, and $R_5(x)$ are all in $\mathbb{T}(u)$.*

In the next five results, it is supposed that u is transitive.

2·5 LEMMA *For x, y in u, $R_4(x,y) = x \times y \subseteq u \times u \subseteq \mathbb{T}^2(u)$.*

2·6 COROLLARY *For x, y in u, $R_4(x,y) \in \mathbb{T}^3(u)$.*

2·7 LEMMA *For a, b, c in u, $(a,c)_2 \in \mathbb{T}^2(u)$ and $(b,a,c)_3 \in \mathbb{T}^4(u)$.*

2·8 COROLLARY *For $x \in u$, $R_6(x)$ and $R_7(x)$ are in $\mathbb{T}^5(u)$.*

2·9 LEMMA *For x, $y \in u$, $R_8(x,y) \in \mathbb{T}^2(u)$.*

Proof : For x, w in u, $x``w \in \mathbb{T}(u)$, so $R_8(x,y) = \mathbb{T}(u) \cap \{x``w \mid_w w \in y\}$; $x, y \in \mathbb{T}(u)$, so $R_8(x,y) \in \mathbb{T}^2(u)$. ⊣ (2·9)

Those remarks, which were proved in *Weak Systems*, though regrettably without the requirement that u be transitive being clearly stated, and of which more general forms will be proved below, immediately yield:

2·10 PROPOSITION *If $F(\vec{x})$ is a rudimentary function of several variables, there is an $\ell \in \omega$ such that for all transitive u, if each argument in $\vec{x}$ is in u, then $F(\vec{x}) \in \mathbb{T}^\ell(u)$.*

Proof : The stated property holds of the nine generating functions and is preserved under composition. ⊣ (2·10)

REMARK **That result may be interpreted as quantifying over programs for rud functions, and thus regarded as a single theorem, not a scheme.**

2·11 COROLLARY (Gandy; Jensen) *If F is rudimentary, then there is a finite ℓ such that the rank of the value is at most the maximum of the ranks of the arguments, plus ℓ.*

Proof : the function $\mathbb{T}$ increases rank by exactly 1. ⊣ (2·11)

2·12 COROLLARY *For any transitive u, $\bigcup_{n\in\omega}\mathbb{T}^n(u)$ is the rudimentary closure of $u \cup \{u\}$ and models* TCo.

The intransitive case

The function $\mathbb{T}$ works very happily for transitive argument, but for intransitive argument it starts to create non-trivial problems. The aim, in the two cases, is not quite the same. The purpose of $\mathbb{T}$ is to proceed by rud steps from any transitive set u to rud(u), which will be of strictly greater rank; with an intransitive argument of limit rank, our first concern would be to fatten it to a transitive rud closed set, without raising rank. Here are two ways of doing so.

We introduce two functions, trud and krud.

2·13 DEFINITION $\mathrm{trud}(u) =_{\mathrm{df}} \bigcup\{F(\vec{x}) \mid F \text{ rud } \& \ \vec{x} \in u\}$.

That is a legitimate definition because we are quantifying over programs for rud functions; the axiom of infinity is at work here. Here $\vec{x}$ denotes a finite sequence of arguments of F, and we follow Devlin's convention that $\vec{x} \in u$ means that each argument is in u; if we wanted to say that the sequence is in u we would write $\langle\vec{x}\rangle \in u$.

2·14 PROPOSITION *For any set u, $\mathrm{trud}(u)$ is transitive, rud closed and includes u; and if A is transitive, rud closed and includes u, then $\mathrm{trud}(u) \subseteq A$. The rank of $\mathrm{trud}(u)$ will be the least limit ordinal greater than or equal to the rank of u.*

Proof : If $a \in b \in F(\vec{x})$, then $a \in \bigcup F(\vec{x}) \subseteq \mathrm{trud}(u)$, $\bigcup \circ F$ being rud; and so $\mathrm{trud}(u)$ is transitive.

If $G(\cdot,\cdot)$ is rud, $b_1 \in F_1(\vec{x})$, $b_2 \in F_2(\vec{y})$, then $G(b_1, b_2) \in G``H(\vec{x},\vec{y})$ for some rud $H$; $G`` \circ H$ is rud, and so $G(b_1, b_2) \in \mathrm{trud}(u)$. Similarly for functions of a different number of variables.

If $a \in u$ then $a \in \{a\} \subseteq \mathrm{trud}(u)$.

If $\vec{x} \in u$ then $\vec{x} \in A$ as A includes u; then $F(\vec{x}) \in A$, A being rud closed; so $F(\vec{x}) \subseteq A$, as A is transitive. Thus $\mathrm{trud}(u) \subseteq A$. ⊣ (2·14)

The definition of trud can be given recursively.

2·15 DEFINITION $\mathbb{K}(u) = u \cup \bigcup u \cup \{R_i(x,y,z) \mid 0 \leqslant i \leqslant 8 \ \& \ x,y,z \in u \cup \bigcup u\}$.

That definition is intended for use even when u is intransitive. Note that $\mathbb{K}$ is rudimentary, and that it has the agreeable property that $u \subseteq v \implies \mathbb{K}(u) \subseteq \mathbb{K}(v)$.

2·16 DEFINITION $\mathbb{K}_0(u) = u$; $\quad \mathbb{K}_{n+1}(u) = \mathbb{K}(\mathbb{K}_n(u))$; $\quad \mathrm{krud}(u) = \bigcup_{n\in\omega}\mathbb{K}_n(u)$.

2·17 PROPOSITION *For any u,* $\mathrm{krud}(u) = \mathrm{trud}(u)$.

Proof : plainly $\mathrm{krud}(u)$ includes u, is transitive and is rud closed; so $\mathrm{trud}(u) \subseteq \mathrm{krud}(u)$.

If $u \subseteq A$ where A is transitive, rud closed and includes u then one verifies by an easy induction that each $\mathbb{K}_n(u) \subseteq A$. Hence $\mathrm{krud}(u) \subseteq \mathrm{trud}(u)$. ⊣ (2·17)

2·18 REMARK $\mathbb{K}$ has the property that for any rud function R there is a d such that $\mathbb{K}^d$ is a 1-companion of R.

Gandy reproved

There are some errors in Gandy's paper *Set theoretic functions* which may have resulted from Gandy encountering similar difficulties to those created by "the intransitive case".

The first such mistake is in his Lemma 1.5.3. on page 111. Start from his definition 1.5.2: he uses a bold-face $\mathbf{x}$ to denote the (meta) finite sequence $x_1, \ldots x_m$: cf the bottom of page 105. I am not sure that this usage is entirely unambiguous; I believe the letter m here to be a variable of the meta-language.

Let us for simplicity take the case $m = 1$, and write x for x_1. Then the first part of his Definition 1.5.2 runs

$$\mathrm{Cc}_0\{x\} = \{x\}; \mathrm{Cc}_{q+1}\{x\} = Cc_q\{x\} \cup \{\mathrm{Cc}_q\{x\}\} \cup \{\mathbf{F}_i uv : 1 \leqslant i \leqslant 9 \ \&\ u, v \in \mathrm{Cc}_q\{x\}\}.$$

I believe that the letter q here is a variable of the language of discourse.

2·19 PROPOSITION *For any $q \in \omega$ and any x, $\mathrm{Cc}_q\{x\}$ is a finite set.*

Proof: by induction on q. Indeed, for a given x, let n_q be the number of elements in $\mathrm{Cc}_q\{x\}$. Then $n_0 = 1$; $n_{q+1} \leqslant n_q + 1 + 9 \cdot {n_q}^2$. ⊣ (2·19)

2·20 Thus the second statement of part (ii) of his Lemma 1.5.3 is false: if x is actually an infinite set, it cannot be a subset of any $\mathrm{Cc}_q\{x\}$.

Gandy defines $\mathrm{Cc}\{x\}$ as $\bigcup_{q\in\omega} \mathrm{Cc}_q\{x\}$; which will be a countable infinite set; so if x is uncountable, it cannot be a subset of $\mathrm{Cc}\{x\}$, even if it is transitive.

Gandy's Lemma 1.5.4 is false; step (C) in the proof he gives is false. Theorem 1.5.5 is half true: the "only if" is false, the proof using the false Lemma 1.5.4. Theorem 1.5.6 is false: $\mathrm{Bc}\{x\}$ is always transitive but $\mathrm{Cc}\{x\}$ need not be. On page 113 the factor 9 is mysteriously omitted from equation (2).

He goes on to state, in [G, pp. 113–4], two interesting and correct results, given below as Propositions 2·23 and 2·25, of which his own proofs are flawed by the above mistakes.

2·21 LEMMA *If u is a finite transitive set with $\overline{\overline{u}} = \ell$, then $\overline{\overline{\mathbb{T}(u)}} \leqslant \frac{1}{2}(2 + 13\ell + 9\ell^2)$.*

Proof: by inspection. ⊣ (2·21)

2·22 DEFINITION (Gandy) $\eta(x) =_{\mathrm{df}}$ the cardinal of the transitive closure of x.

2·23 PROPOSITION (Gandy) *If F is rud, then there is a k such that $\eta(F(\vec{x}))$ is less than $(\eta(\{\vec{x}\}) + 1)^k$.*

Here $\{\vec{x}\}$ for many variables means the set of them.

Proof: we know that there is an ℓ such that for u transitive and the arguments of F in u, $F(\vec{x}) \in \mathbb{T}^\ell(u)$. For u transitive, $\mathbb{T}(u)$ is transitive, and iterating the previous estimate, we find that there is a polynomial $Q(X)$ of degree 2^ℓ, (for example $13^{2^\ell - 1}X^{2^\ell}$) such that $x \in u$ implies that $\eta(F(\vec{x}))$ is at most $Q(\overline{\overline{u}} + 1)$. ⊣ (2·23)

2·24 REMARK We may now justify our earlier remark that there is no pure rud recursion for $\mathcal{S}(x)$ for x an arbitrary set. If we look at $\mathcal{S}(x)$ for $x \in HF$, we see that $\mathcal{S}(V_n) = V_{n+1}$; if $\mathcal{S}(x)$ were pure rud rec, given by G, we would have

$$G(\mathcal{S} \restriction V_n) = V_{n+1}.$$

But if $\overline{\overline{V_n}} = N$, $\overline{\overline{V_{n+1}}} = 2^N$, whereas

$$\mathrm{tcl}(\mathcal{S} \restriction V_n) \subseteq \{(\mathcal{S}(x), x) \mid x \in V_n\} \cup \{\{\mathcal{S}(x)\} \mid x \in V_n\} \cup \{\{\mathcal{S}(x), x\} \mid x \in V_n\} \cup \{\mathcal{S}(x) \mid x \in V_n\} \cup V_n$$

which has cardinality at most $5N$; but for each k, $(5N)^k$ will be much less than 2^N for large N. ⊣ (2·24)

Gandy remarks on page 114 that there is a primitive recursive function which returns the value ω given any argument of infinite rank. Indeed the example he gives is rud rec: define

$$F(x) = \omega \cap \bigcup \{F(y) \cup \{F(y)\} \mid_y y \in x\},$$

which is rud rec as intersection with ω is given by a Δ_0 separator; and show first that if $x \in \mathbf{HF}$, then $F(x) = \varrho(x)$.

He then states as his Theorem 2.1.3 the following:

2·25 PROPOSITION (Gandy) *There is a set c of infinite rank such that for no rud function G is $G(c) = \omega$.*

His proof of that is invalidated by his use of the false Theorem 1.5.6. The result though is interesting and is correct, since it is possible to build a transitive model of Z not containing ω but containing sets of infinite rank. Such models are automatically rud closed, and absolute for rud functions. These constructions make heavy use of the power set axiom, and accordingly we place the details in a separate paper, [M5].

3: The class of pure rud rec functions

3·0 By type I or pure rudimentary recursion we mean those given by a recursion equation of the form

$$F(x) = G(F \restriction x)$$

where G is a pure rud function with no hidden parameters.

3·1 Proposition *Every (unary) rud function is rud rec.*

Proof : If $F(\cdot)$ is unary and rud, let $G(f) =_{\mathrm{df}} F(\mathrm{Dom}\,(f))$; then G is rud and $\forall x\; F(x) = G(F \restriction x)$. Other rud functions can be transformed to unary functions by using the pairing and un-pairing functions, which are rudimentary. ⊣ (3·1)

3·2 Proposition *If F_1 and F_2 are rud rec, so is $x \mapsto (F_1(x), F_2(x))_2$.*

Proof : Let $K(x) = \big(F_1(x), F_2(x)\big)_2$. Then $K(x) = \big(G_1(F_1 \restriction x), G_2(F_2 \restriction x)\big)_2$.

$K \restriction x = \{\big((F_1(a), F_2(a))_2, a\big)_2 \mid_a a \in x\}$.

There are rud G_3 and G_4 such that $G_3(K \restriction x) = F_1 \restriction x$ and $G_4(K \restriction x) = F_2 \restriction x$. So

$$K(x) = \big(G_1(G_3(K \restriction x)), G_2(G_4(K \restriction x))\big)_2 = G_5(K \restriction x)$$

where $G_5(z) =_{\mathrm{df}} \big(G_1(G_3(z)), G_2(G_4(z))\big)_2$. G_5 is rudimentary. ⊣ (3·2)

3·3 Proposition *Let G_1 and G_2 be rudimentary, and suppose that F_1 and F_2 are defined by the simultaneous recursion*

$$F_1(x) = G_1(F_2 \restriction x); \qquad F_2(x) = G_2(F_1 \restriction x).$$

Then both F_1 and F_2 are projections of a rud rec function.

Proof : Let $K(x) = \big(F_1(x), F_2(x)\big)_2$. Then $K(x) = (G_1(F_2 \restriction x), G_2(F_1 \restriction x))_2$.

$K \restriction x = \{\big((F_1(a), F_2(a))_2, a\big)_2 \mid_a a \in x\}$.

There are rud G_3 and G_4 such that $G_3(K \restriction x) = F_1 \restriction x$ and $G_4(K \restriction x) = F_2 \restriction x$. So

$$K(x) = \big(G_1(G_4(K \restriction x)), G_2(G_3(K \restriction x))\big)_2 = G_6(K \restriction x)$$

where $G_6(z) =_{\mathrm{df}} \big(G_1(G_4(z)), G_2(G_3(z))\big)_2$. G_6 is rudimentary. ⊣ (3·3)

3·4 Proposition *If F is rud rec, so is $x \mapsto F \restriction x$.*

Proof : Let F be given by G, and let $H(x) = F \restriction x$. Then

$$\begin{aligned} H(x) &= F \restriction x \\ &= \{(F(a), a)_2 \mid_a a \in x\} \\ &= \{\big(G(F \restriction a), a\big)_2 \mid_a a \in x\} \\ &= \{\big(G(H(a)), a\big)_2 \mid_a a \in x\} \\ &= G_2(H \restriction x) \end{aligned}$$

where, setting G_1 to be the rud function $x \mapsto \big(G(\mathrm{left}(x)), \mathrm{right}(x)\big)_2$, we take $G_2(x) =_{\mathrm{df}} G_1``x$. ⊣ (3·4)

3·5 Corollary *Thus $F``$, being equal to $Im \circ (F \restriction)$, is rud of rud rec.*

3·6 Remark Here is a case when "rud of rud rec" is rud rec. Let $F(x) = G_1(F \restriction x)$ and $H(x) = G_2(F(x))$ where G_1 and G_2 are rud. Suppose that there is a rud function G_3 such that for all x, $G_3(G_2(x)) = x$. Then there is a rud G_4 such that $F \restriction x = G_4(H \restriction x)$, since $H \restriction x = \{\big(G_2(F(y)), y\big)_2 \mid_y y \in x\}$ and $F \restriction x = \{(F(y), y)_2 \mid_y y \in x\}$. So

$$H(x) = G_2(G_1(G_4(H \restriction x))),$$

and is thus rud rec.

That argument will also work when the hypothesis on G_3 only holds for certain x.

3·7 Example Let $F(x) = \mathrm{tcl}(x)$, which we know to be rud recursive: $F(x) = G(F \restriction x)$. Here $F(x)$ is always a transitive set; for u transitive, $\bigcup \mathbb{T}(u) = u$. So if we set $H(x) = \mathbb{T}(\mathrm{tcl}(x))$, the above argument will show that H is rud rec.

Recursion on related relations

3·8 PROPOSITION *Let F be defined by $F(x) = G(x, F\restriction \mathrm{tcl}(x))$, where G is rudimentary. Define H by $H(x) = F\restriction \mathrm{tcl}(\{x\})$. Then H is rud rec and therefore F is rud rud rec.*

Proof : $F\restriction \mathrm{tcl}(x) = \bigcup_{y\in x} H(y)$, so

$$\begin{aligned} H(x) &= \{(F(x), x)_2\} \cup \bigcup_{y\in x} H(y) \\ &= \{\big(G(x, \bigcup_{y\in x} H(y)), x\big)_2\} \cup \bigcup_{y\in x} H(y) \\ &= G_1(H\restriction x) \end{aligned}$$

where $G_1(h) = \{\big(G(\mathrm{Dom}\,(h), \bigcup \mathrm{Im}\,(h)), \mathrm{Dom}\,(h)\big)_2\} \cup \bigcup \mathrm{Im}\,(h)$, so that G_1 is rudimentary and H is rud rec. Then $F(x) = [H(x)](x)$, the evaluation of $H(x)$ at argument x, and is thus a trivial rud function of x and $H(x)$. ⊣ (3·8)

3·9 COROLLARY *Functions defined by recursions of the form $F(x) = G(x, F\restriction \bigcup\bigcup x)$ are thus rud of rud rec.*

3·10 REMARK Recursions of that kind occur in the definition of forcing.

3·11 REMARK Let $a = \mathrm{tcl}(\{x\})$; then $\bigcup a = \mathrm{tcl}(x)$, $\{x\} = a \setminus \bigcup a$ and $x = \bigcup(a \setminus \bigcup a)$.

3·12 PROPOSITION *If $\forall x\, F(x) = G(F\restriction x)$ with G rud there is a rud $G_1(\cdot,\cdot)$ such that*

$$\forall x\, F(x) = G_1(x, F\restriction \mathrm{tcl}(x)).$$

Proof : Take $G_1(x, f) = G(f\restriction x)$. ⊣ (3·12)

3·13 COROLLARY *If F is rud rec, and we put $H(x) = F\restriction \mathrm{tcl}(\{x\})$, then H is rud rec.*

Proof : By 3·8 and 3·12.

An illusory recursion

Just to warn the reader:

3·14 PROPOSITION *There are rud functions G and H such that for any function F, $F(x) = G(F\restriction H(x))$.*

Rud rec of rud is rud of rud rec

3·15 LEMMA *Let F be rud rec, given by $F(x) = G(F\restriction x)$ where G is rud. Then there is a rud function H obtainable uniformly from G such that for every transitive u, $F\restriction \mathbb{T}(u) = H(F\restriction u)$, $F\restriction \mathbb{T}^2(u) = H^2(F\restriction u)$, and more generally for each positive ℓ, $F\restriction \mathbb{T}^\ell(u) = H^\ell(F\restriction u)$.*

Proof : For transitive u, $\mathbb{T}(u)$ is a collection of subsets of u, so for $x \in \mathbb{T}(u)$, $F\restriction x = (F\restriction u)\restriction x$. Let $\phi(f, x) = (G(f\restriction x), x)_2$. Then ϕ is rud, and

$$F\restriction \mathbb{T}(u) = \{\phi(F\restriction u, x) \,|_x\, x \in \mathbb{T}(u)\} = H(F\restriction u)$$

where H is rud.

Then $F\restriction \mathbb{T}^2(u) = H(F\restriction \mathbb{T}(u)) = H^2(F\restriction u)$, and an induction will now apply. ⊣ (3·15)

3·16 LEMMA *For i = 1, 2, Let F_i be rud of rud rec. Then the function $x \mapsto \big(F_1(x), F_2(x)\big)_2$ is rud of rud rec.*

Proof : let $F_i(x) = R_i(H_i(x))$ where R_i is rud and H_i is rud rec. By Proposition 3·2, $x \mapsto \big(H_1(x), H_2(x)\big)_2$ is rud rec, and $(a, b) \mapsto \big(R_1(a), R_2(b)\big)_2$ is rudimentary. ⊣ (3·16)

3·17 PROPOSITION *Let F_1 be rud rec, given by G_1 and let F_2 be rud. Then the function $x \mapsto F_1(F_2(x))$ is rud of rud rec.*

Proof : Fix $\ell < \omega$ such that for all x, $F_2(x) \in \mathbb{T}^\ell(\mathrm{tcl}(\{x\}))$. Such ℓ exists by Proposition 2·12. Then there are rud functions G_3 and H such that

$$\begin{aligned} F_1(F_2(x)) &= G_1(F_1 \restriction F_2(x)) \\ &= G_3(x, F_1 \restriction \mathbb{T}^\ell(\mathrm{tcl}(\{x\})), F_2(x)) \\ &= G_3(x, H^\ell(F_1 \restriction \mathrm{tcl}(\{x\})), F_2(x)) && \text{by Lemma 3·13} \end{aligned}$$

which is rud of rud rec, by Corollary 3·13 and by hypothesis. ⊣ (3·17)

Rud of rud rec is projection of rud rec

3·18 PROPOSITION *Let $E(x) = H(F(x))$ where F is rud rec, given by G_1 and H is rud. Let $K(x) = (E(x), F(x))$. Then K is rud rec, and $E(x) = left((K(x)))$.*

Proof : Let G_2 be a rud function such that $G_2(K \restriction x) = F \restriction x$: as $K \restriction x = \{\big((E(a), F(a))_2, a\big)_2 \mid a \in x\}$, such G_2 exists.

As $K(x) = \big(H(G_1(F \restriction x)), G_1(F \restriction x)\big)_2$, we have $K(x) = G_3(K \restriction x)$, where

$$G_3(f) \ =_{\mathrm{df}} \ \big(H(G_1(G_2(f))), G_1(G_2(f))\big)_2. \qquad ⊣ (3·18)$$

4: Rudimentary recursion from parameters

4·0 We have defined functions of type I, or pure rud rec functions to be those given by a recursion equation of the form

$$F(x) = G(F \restriction x)$$

where G is a pure rud function with no hidden parameters.

4·1 For recursions involving parameters, the following definition seems the most satisfactory, which we call type II.

$$F(x) = G(p, F \restriction x)$$

Here G is a pure rud function of two variables and p is some set. We shall call such an F *p-rud rec* or *a function of Type II.*

4·2 REMARK An earlier formulation was the following, which we call type II':

$$F(x) = \begin{cases} a & \text{if } x = \varnothing; \\ G(F \restriction x) & \text{if } x \neq \varnothing. \end{cases}$$

Here G is a pure rud function of one variable, and a is some set.

For example, the function $\nu \mapsto \omega + \nu$ is of type II'.

Every function of type II' is expressible in form II. But the proof of the important Proposition 3·4, that if F is rud rec, so is $x \mapsto F \restriction x$, appears not to adapt to type II', but does adapt to type II. Therefore we favour type II. Proposition 4·10 will say that if F is p-rud rec, so is $x \mapsto F \restriction x$.

4·3 REMARK Type II will underlie our discussion of rudimentary forcing, with the poset $\mathbb{P}$ of conditions as an ever-present parameter.

REMARK The first Jensen fragment after J_1 that is closed under functions of Type II' is J_ω, as given J_k we could set $f(0) = J_k; f(n+1) = \mathbb{T}(f(n)); f(\lambda) = \bigcup f``\lambda$, and then $f(\omega) = J_{k+1}$.

4·4 Finally we turn the parameter back into a variable by considering recursion equations of the following form, which we shall call type III, though in this paper we shall say little about them.

$$F(v, x) = G(v, F \restriction (\{v\} \times x)).$$

4·5 Remark The recursion here is on the second variable, in harmony with the form of the definition of ordinal addition as given in Example 0·12.

4·6 Proposition *For each fixed v the map $x \mapsto F(v, x)$ is rud recursive of type II, in the parameter v.*

Proof : Let $E(x) = F(v, x)$. Then $E \restriction x = \{(F(v, b), b)_2 \mid_b b \in x\}$ whereas

$$\begin{aligned} F \restriction (\{v\} \times x) &= \{\big(F(v, b), (v, b)_2\big)_2 \mid_b b \in x\} \\ &= \{\big(E(b), (v, b)_2\big)_2 \mid_b b \in x\} \\ &= H(v, E \restriction x) \end{aligned}$$

for a certain rud function H; so $E(x) = G(v, H(v, E \restriction x)) = G_1(v, E \restriction x)$, for some rud function G_1. ⊣ (4·6)

4·7 Remark x seems to be rud recoverable from $F \restriction (\{v\} \times x)$ as the domain of its domain. So at present I think nothing of substance is to be gained by considering equations of the form

$$F(v, x) = H(v, x, F \restriction (\{v\} \times x)).$$

4·8 Remark A difference between Type II and Type III is that, as shown by Proposition 5·44, the J hierarchy after a good start for a type II recursion will continue to support that recursion at every step thereafter, but only supports type III recursions at indecomposable stages.

Some closure properties

4·9 Proposition *If F is p-rud-rec, so is $x \mapsto F \restriction x$.*

Proof : Suppose that for all x, $F(x) = G(p, F \restriction x)$, where G is rud, and let $H(x) = F \restriction x$. Then

$$\begin{aligned} H(x) &= F \restriction x \\ &= \{(F(a), a)_2 \mid_a a \in x\} \\ &= \{(G(p, F \restriction a), a)_2 \mid_a a \in x\} \\ &= \{(G(p, H(a)), a)_2 \mid_a a \in x\} \\ &= G_4(p, H \restriction x) \end{aligned}$$

where $G_4(y, f) = \{(G(y, d), a)_2 \mid_{d,a} (d,\, a) \in f\}$; G_4 is G_3“ for some rud G_3. ⊣

4·10 Proposition *Let F be rud rec, given by $F(x) = G(p, F \restriction x)$ where G is rud. Then there is a rud function H obtainable uniformly from G such that for every transitive u, $F \restriction \mathbb{T}(u) = H(p, F \restriction u)$;*

Proof : For transitive u, $\mathbb{T}(u)$ is a collection of subsets of u, so for $x \in \mathbb{T}(u)$, $F \restriction x = (F \restriction u) \restriction x$. This time let $\phi(z, f, x) = (G(z, f \restriction x), x)$. Then ϕ is rud, and

$$F \restriction \mathbb{T}(u) = \{\phi(p, F \restriction u, x) \mid_x x \in \mathbb{T}(u)\} = H(p, F \restriction u)$$

where H is again rud. ⊣

The iteration is not so convenient as in the pure case, since $F \restriction \mathbb{T}^2(u) = H(p, H(p, F \restriction u))$; so it is better to put it this way:

4·11 Proposition *There is a rudimentary K such that*

$$(p, F \restriction \mathbb{T}(u))_2 = K(p, F \restriction u),\ (p, F \restriction \mathbb{T}^2(u))_2 = K^2(p, F \restriction u),$$

and more generally for each positive ℓ,

$$(p, F \restriction \mathbb{T}^\ell(u))_2 = K^\ell(p, F \restriction u).$$

5: Provident sets

5·0 DEFINITION A set A is *p-provident*, where p is a set, if it is non-empty, transitive, closed under pairing and for all p-rud rec F and all x in A, $F(x) \in A$.

5·1 REMARK If A is p-provident, $p \in A$.

5·2 EXAMPLE We shall see that the Jensen fragment J_ν is $\varnothing$-provident for all $\nu \geqslant 1$.

5·3 DEFINITION A is *provident* if it is p-provident for every $p \in A$.

5·4 REMARK The only provident set not containing an infinite set is **HF**.

5·5 REMARK For provident sets, it is unnecessary to demand that they be closed under pairing, for if $x \in A$, the function $y \mapsto \{x, y\}$ is x-rud rec, being given by the recursion $F(y) = \{x, \mathrm{Dom}\, F \restriction y\}$. But the union of two sets each closed under $\varnothing$-rud rec functions might not be closed under pairing, though as rud rec functions are unary, that union would be closed under $\varnothing$-rud rec functions: for example, let a and b be mutually Cohen-generic subsets of ω and consider the model $J_2(a) \cup J_2(b)$.

A typical provident set is $J_{\omega^\nu}(a)$ provided ω^ν is greater than the rank of the transitive set a. But it proves desirable to alter the customary definition of $L(a)$ mentioned in Example 0·6.

Bounding rudimentary functions in a finite progress

5·6 DEFINITION Let ξ be an ordinal. A *ξ-progress* is a sequence $\langle P_\nu \mid \nu \leqslant \xi\rangle$ of transitive sets such that for each $\nu < \xi$, $\mathbb{T}(P_\nu) \subseteq P_{\nu+1}$ and for each limit ordinal $\lambda \leqslant \xi$, $\bigcup_{\nu<\lambda} P_\nu \subseteq P_\lambda$; the progress is *strict* if for each $\nu < \xi$, $P_{\nu+1} \subseteq \mathcal{P}(P_\nu)$; and *continuous* if for each limit $\lambda \leqslant \xi$, $P_\lambda = \bigcup_{\nu<\lambda} P_\nu$.

5·7 PROPOSITION *If the progress is strict and continuous then for each $\nu \leqslant \xi$, $\varrho(P_\nu) = \varrho(P_0) + \nu$.*

Proof : for transitive u, $\varrho(\mathbb{T}(u)) = \varrho(u) + 1 = \varrho(\mathcal{P}(u))$. ⊣ (5·7)

5·8 THEOREM *Let R be a rudimentary function of n variables. There is a $c_R \in \omega$ such that for every c_R-progress $P_0, P_1, \ldots, P_{c_R}$, $R``P_0^n \subseteq P_{c_R}$.*

5·9 DEFINITION We call c_R the *rudimentary constant* of R.

5·10 REMARK More precisely, there is a recursive function sending a program for R to a bound; but the function sending a program for R to the minimal bound is not recursive.

We prove the theorem in a series of lemmata.

5·11 LEMMA *If x and y are in P_ν then $\{x, y\} \in P_{\nu+1}$, $x \smallsetminus y \in P_{\nu+1}$, $\bigcup x \in P_{\nu+1}$ and* $\mathrm{Dom}\,(x) \in P_{\nu+1}$.

Proof : Immediate from lines 2, 3, 4 and 5 of the definition of $\mathbb{T}$. ⊣ (5·11)

5·12 LEMMA $x, y \in P_\zeta \implies x \times y \in P_{\zeta+3}$.

Proof : If x and y are in P_ν then both $\{x\}$ and $\{x, y\}$ are in $P_{\nu+1}$; so $\{\{x\}, \{x, y\}\}$ are in $P_{\nu+2}$; P_ν being transitive, we may infer that if $a \in x$ and $b \in x$, then $(a, b)_2$ is in $P_{\nu+2}$; thus $x \times y \subseteq P_{\nu+2}$, which, since $P_\nu \subseteq P_{\nu+2}$, implies that $x \times y \in P_{\nu+3}$. ⊣ (5·12)

5·13 LEMMA $x, y \in P_\zeta \implies R_5(x, y) \in P_{\zeta+1}$.

5·14 LEMMA $a, b, c \in P_\zeta \implies [(a, c)_2 \in P_{\zeta+2}\ \&\ (b, a, c)_3 \in P_{\zeta+4}]$.

5·15 LEMMA $x \in P_\zeta \implies R_6(x) \in P_{\zeta+5}$.

5·16 LEMMA $x \in P_\zeta \implies R_7(x) \in P_{\zeta+5}$.

5·17 LEMMA $x, w \in P_\zeta \implies x``\{w\} \in P_{\zeta+1}$.

5·18 LEMMA $x, y \in P_\zeta \implies R_8(x, y) \in P_{\zeta+2}$.

Proof of Theorem 5·8: The lemmata show that for $i = 0, \ldots 8$, we may take c_{R_i} to be 1, 1, 1, 1, 3, 1, 5, 5, 2 respectively. The theorem now follows by remarking that if S and T_i are rudimentary and for all x, $Q(\vec{x}) = S(T_0(\vec{x}), \ldots, T_k(\vec{x}))$, we may take $c_Q = c_S + \max_i c_{T_i}$. ⊣ (5·8)

5·19 COROLLARY *If $\langle P_\nu \mid \nu \leqslant \xi\rangle$ is a ξ-progress, then at each limit ordinal $\lambda \leqslant \xi$, $\bigcup_{\nu<\lambda} P_\nu$ is rud closed.*

$\dot{\Delta}_0$ separators

5·20 Remark If φ is $\dot{\Delta}_0$ the function $a \mapsto a \cap \{x \mid \overset{0}{\models} \varphi(x, b)\}$ is basic; in this case we speak of the *separational delay*. Here $\overset{0}{\models}$ is the truth definition for $\dot{\Delta}_0$ wffs discussed in paragraph 5·7 of [M2] and in Appendix Three.

The canonical progress towards a given transitive set

5·21 Let c be a transitive set. Let $c_\zeta = c \cap \{x \mid \varrho(x) < \zeta\}$. Since c is transitive, $c_{\zeta+1}$ will be a set of subsets of c_ζ; in fact $c_{\zeta+1} = c \cap \{x \mid x \subseteq c_\zeta\}$; we shall use this as a direct recursive definition below.

If $c_{\zeta+1} = c_\zeta$, then $c_\zeta = c$ and for all $\xi > \zeta$, $c_\xi = c_\zeta$; so that that first happens when $\zeta = \varrho(c)$.

Using c as a parameter we define a sequence of pairs $((c_\nu, P^c_\nu))_\nu$ by a rud recursion on ν. Each P^c_ν will be of rank ν; we shall use the function $\mathbb{T}$, but we shall also "feed" stages of c into the process.

The sequence $(P^c_\nu)_\nu$ forms a strict continuous progress; such is the importance of this definition in the sequel that we shall call this the *canonical progress* towards, to, or through c, the choice of preposition depending on the length of the sequence as compared to the rank of c.

5·22 Definition

$$\begin{array}{lllllllll} c_0 & = & \varnothing & c_{\nu+1} & = & c \cap \{x \mid x \subseteq c_\nu\} & c_\lambda & = & \bigcup_{\nu<\lambda} c_\nu \\ P^c_0 & = & \varnothing & P^c_{\nu+1} & = & \mathbb{T}(P^c_\nu) \cup \{c_\nu\} \cup c_{\nu+1} & P^c_\lambda & = & \bigcup_{\nu<\lambda} P^c_\nu \end{array}$$

5·23 Lemma *Each P^c_ν is transitive. $P^c_\nu \subseteq P^c_{\nu+1}$. $P^c_\nu \in P^c_{\nu+1}$; and so for $\nu < \zeta$, $P^c_\nu \subseteq P^c_\zeta$ and $P^c_\nu \in P^c_\zeta$.*

5·24 Remark $c_\nu = c \cap P^c_\nu$; $\varrho(P^c_\nu) = \nu$.

5·25 Remark P^c_ν may be defined by a single rud recursion on ordinals:

$$P^c_0 = \varnothing; \qquad P^c_{\nu+1} = \mathbb{T}(P^c_\nu) \cup \{c \cap P^c_\nu\} \cup (c \cap \{x \mid x \subseteq P^c_\nu\}); \qquad P^c_\lambda = \bigcup_{\nu<\lambda} P^c_\nu.$$

With that definition, one should then verify by induction that for each ν, $c \cap P^c_\nu = c \cap \{x \mid \varrho(x) < \nu\}$, and thence that the two definitions agree.

5·26 Remark Each P^c_λ is rud closed, for λ a limit ordinal, by Theorem 5·7.

5·27 Remark $P^c_\omega = V_\omega$: for each $P^c_n \subseteq V_n$ and so $P^c_\omega \subseteq V_\omega$; equality will follow from the fact that P^c_ω is a non-empty rud closed set, by the previous remark.

Two important properties of p-rud rec functions.

5·28 The Definability Lemma *Let F be p-rud recursive, given by G. Then "f is an F-attempt" is a Δ_0 predicate of p and f.*

Proof : The predicate is $Fn(f)$ & $\bigcup \mathrm{Dom}\,(f) \subseteq \mathrm{Dom}\,(f)$ & $\forall x{:}{\in} \mathrm{Dom}\,(f)\;\; f(x) = G(p, f \restriction x)$. ⊣ (5·28)

Note Propositions 3·4 and 4·10, which say that if F is rud rec, (in a parameter) , so is $x \mapsto F \restriction x$ (in the same parameter).

5·29 The Propagation Lemma *Let G be a binary rudimentary function. Then there is a ternary rudimentary function H_G, obtainable uniformly from G, such that for any set p, if F be the p-rud rec function given by the recursion $F(x) = G(p, F \restriction x)$, and if P^+ and P be transitive sets with $P \subseteq P^+ \subseteq \mathcal{P}(P)$, then*

$$F \restriction P^+ = H_G(p, F \restriction P, P^+).$$

Proof : If $x \in P^+$, then $x \subseteq P$, so $F \restriction x = (F \restriction P) \restriction x$ so $F(x) = G(p, (F \restriction P) \restriction x)$. Hence

$$F \restriction P^+ = \{(G(p, (F \restriction P) \restriction x), x)_2 \mid_x x \in P^+\}.$$

We take $H_G(p, f, q) \equiv \{(G(p, f \restriction x), x)_2 \mid_x x \in q\}$. ⊣ (5·29)

A new symbol

5·30 Definition $F \upharpoonleft\!\upharpoonright u =_{\mathrm{df}} \{F{\upharpoonright}x \mid x \in u\}$

5·31 Lemma *Let F be rud rec, given by G. There is a rudimentary function G_6 such that for every a, $F{\upharpoonright}a = G_6``(F \upharpoonleft\!\upharpoonright a)$.*

Proof :

$$\begin{aligned} F{\upharpoonright}a &= \{(F(b), b)_2 \mid b \in a\} \\ &= \{(G(F{\upharpoonright}b), b)_2 \mid b \in a\} \end{aligned}$$

Take $G_6(f) = (G(f), \mathrm{Dom}\,(f))_2$. Then $F{\upharpoonright}a = G_6``\{F{\upharpoonright}b \mid b \in a\} = G_6``F \upharpoonleft\!\upharpoonright a$. ⊣ (5·31)

5·32 Lemma *There is a rudimentary function G_5 such that for any transitive u, $F \upharpoonleft\!\upharpoonright \mathbb{T}(u) = G_5(F{\upharpoonright}u)$.*

Proof : Note that $x \in \mathbb{T}(u) \implies x \subseteq u$, so for such x, $F{\upharpoonright}x = (F{\upharpoonright}u){\upharpoonright}x$; so

$$F \upharpoonleft\!\upharpoonright \mathbb{T}(u) = \{(F{\upharpoonright}u){\upharpoonright}x \mid x \in \mathbb{T}(u)\}.$$

Let $G_5(f) = \{f{\upharpoonright}x \mid x \in \mathbb{T}(\mathrm{Dom}\,(f))\}$. ⊣ (5·32)

5·33 Remark $F``u = G``F \upharpoonleft\!\upharpoonright u$, so if $F \upharpoonleft\!\upharpoonright u \subseteq v$, then $F``u \subseteq G``v$.

5·34 Remark By Lemma 5·31, $F{\upharpoonright}\mathbb{T}(u) = G_6``(F \upharpoonleft\!\upharpoonright \mathbb{T}(u))$

Bounding rudimentarily recursive functions in a single canonical progress

5·35 Theorem *Let F be p-rud rec, given by G. Then there exist s_F and g_F in ω such that for any transitive c and any ordinal ν_0 with $p \in P^c_{\nu_0}$, any non-successor ordinal λ and any $k \in \omega$,*

$$\textit{(i)}\ F \upharpoonleft\!\upharpoonright P^c_\lambda \subseteq P^c_{\nu_0+\lambda}; \qquad \textit{(ii)}\ F{\upharpoonright}P^c_{\lambda+k} \in P^c_{\nu_0+\lambda+s_F+k\cdot g_F}.$$

Proof : For $\lambda = 0$, (i) is trivially true as for any c, $F \upharpoonleft\!\upharpoonright P^c_0 = \varnothing$.

Suppose that Part (i) is true for a given λ, either 0 or a limit ordinal. We shall prove Part ii) for that λ for all k.

From (i) we know that $F \upharpoonleft\!\upharpoonright P^c_\lambda \subseteq P^c_{\nu_0+\lambda}$. By Lemma 5·28, using the fact that $P^c_{\nu_0+\lambda} \in P^c_{\nu_0+\lambda+1}$ and Remark 5·20, there will be a separational delay s_F, such that for any c, ν_0 and limit λ as above, $F{\upharpoonright}P^c_\lambda \in P^c_{\nu_0+\lambda+s_F}$.

We then use the Propagation Lemma 5·29, which provides a rudimentary function H_G such that for every c, λ and k, $F{\upharpoonright}P^c_{\lambda+k+1} = H_G(p, F{\upharpoonright}P^c_{\lambda+k}, P^c_{\lambda+k+1})$; so if g_F is the rudimentary constant for H_G, (which we shall now call the *generational delay* for F), $F{\upharpoonright}P^c_{\lambda+1}$ will be a member of $P^c_{\lambda+s_F+g_F}$.

Repeating the application of H_G, we see that for each finite k,

$$F{\upharpoonright}P^c_{\lambda+k} \in P^c_{\lambda+s_F+k\cdot g_F}.$$

Remark now that if (ii) holds for a given λ and c for all k then (i) holds for $\lambda+\omega$. Finally, if λ is a limit limit ordinal, and property (i) holds for all smaller limits, then it holds at λ. ⊣ (5·35)

5·36 Theorem *Let θ be indecomposable and c a transitive set. Then P^c_θ is provident.*

Proof : Let $p \in P^c_\theta$; choose $\nu_0 < \theta$ with $p \in P^c_{\nu_0}$. Let F be p-rud rec. Then for each limit $\eta < \theta$, $F \upharpoonleft\!\upharpoonright P^c_\eta \subseteq P^c_{\nu_0+\eta} \subseteq P^c_\theta$. So $F \upharpoonleft\!\upharpoonright P^c_\theta \subseteq P^c_\theta$, as required. ⊣ (5·36)

5·37 Proposition *Let c be a transitive set and θ an indecomposable ordinal. Then*

$$P^c_\theta = P^{c_\theta}_\theta = \bigcup_{\lambda<\theta} P^{c_\lambda}_\theta.$$

Proof : If $x \in P^c_\theta$, then for some $\lambda < \theta$, $x \in P^c_\lambda = P^{c_\lambda}_\lambda \subseteq P^{c_\lambda}_\theta$.

Conversely, if $\lambda < \theta$, c_λ is in P^c_θ, which we know to be provident, and the map $\nu \mapsto P^{c_\lambda}_\nu$ is given by a c_λ-rudimentary recursion, and so each $P^{c_\lambda}_\nu$, for $\nu < \theta$, is in P^c_θ; thus $P^{c_\lambda}_\theta \subseteq P^c_\theta$. ⊣ (5·37)

Functions of uniformly affine delay

5·38 DEFINITION We shall say that a unary (class) function F is of *uniformly affine delay* if there are a parameter p and natural numbers s_F, g_F such that for all transitive c containing p, all non-successor ordinals λ and all natural numbers k,

$$F \restriction P^c_{\lambda+k} \in P^c_{\varrho(p)+\lambda+s_F+kg_F}.$$

Thus we have just proved that every p-rud rec function is of uniformly affine delay.

5·39 LEMMA *For functions f, $(f,g)_2 \mapsto f \circ g$ is rudimentary.*

Proof : $f \circ g = (\mathrm{Im}\, f \times \mathrm{Dom}\, g) \cap \{(c,a)_2 \mid \exists b{:}\in\bigcup^2 g\ (b,a)_2 \in g\ \&\ (c,b)_2 \in f\}$. ⊣ (5·39)

5·40 DEFINITION Let $c_\circ$ be the rudimentary constant of the above binary function. ⊣ (5·40)

5·41 THEOREM *If F_1 and F_2 are both unary functions of uniformly affine delay, so is their composition $F_2 \circ F_1$.*

Proof : Let $F_3 = F_2 \circ F_1$ and for i = 1, 2, let p_i, s_i and g_i be the parameters and natural numbers as in the definition of uniformly affine delay.

Let c be a transitive set of which both p_1 and p_2 are members. Let λ be any non-successor ordinal and $k \in \omega$. Then

$$\begin{aligned} & F_1 \restriction P^c_{\lambda+k} \in P^c_{\varrho(p_1)+\lambda+s_1+kg_1} \\ \text{and} \quad & F_2 \restriction P^c_{\varrho(p_1)+\lambda+s_1+kg_1} \in P^c_{\varrho(p_2)+\varrho(p_1)+\lambda+s_2+(s_1+kg_1)g_2}; \\ \text{but} \quad & F_3 \restriction P^c_{\lambda+k} = \big(F_2 \restriction P^c_{\varrho(p_1)+\lambda+s_1+kg_1}\big) \circ \big(F_1 \restriction P^c_{\lambda+k}\big) \\ \text{so} \quad & F_3 \restriction P^c_{\lambda+k} \in P^c_{\varrho(p_2)+\varrho(p_1)+\lambda+s_2+(s_1+kg_1)g_2+c_\circ}; \\ \text{thus we shall have} \quad & F_3 \restriction P^c_{\lambda+k} \in P^c_{\varrho(p_3)+\lambda+s_3+kg_3} \end{aligned}$$

as required, if we set $s_3 = s_2 + s_1 g_2 + c_\circ$, $g_3 = g_1 g_2$, and p_3 to be some set of rank at least $\varrho(p_2) + \varrho(p_1)$ that includes $\{p_1, p_2\}$. ⊣ (5·41)

5·42 REMARK If $s_1 > 0$, $F_1 \restriction P^c_{\lambda+k} \subseteq P^c_{\varrho(p_1)+\lambda+s_1+kg_1-1}$, so that some improvement in the above constants will be possible.

5·43 REMARK If F is rud rec and E is a unary rudimentary function, then we may take $s_{E\circ F} = s_F + c_E$ and $g_{E\circ F} = g_F$; for in effect $s_E = c_E$ and $g_E = 1$.

5·44 PROPOSITION *Let $\langle P_\nu \mid \nu \leqslant \omega\rangle$ be a strict continuous ω-progress, and let $p \in P_0$ If P_0 is p-provident, so is P_ω.*

Proof : P_ω will be rud closed by Corollary 5·19, and hence closed under pairing.

Let F be p rud rec. Then $F \upharpoonright\!\!\upharpoonright P_0 \subseteq P_0$ by hypothesis, bearing in mind Proposition 4·10. $P_0 \in \mathbb{T}(P_0) \subseteq P_1$, and so $F \restriction P_0$, being by Lemma 5·28 definable over P_0 will be a Δ_0 subset of P_1, and thus in some P_n, n being given by the appropriate separational delay. Lemma 5·29 will continue the propagation, so that each $F \restriction P_n$ will be in P_ω. Hence P_ω will be closed under F. ⊣ (5·44)

Enhanced version

5·45 PROPOSITION *Let c be transitive, ξ a limit ordinal. Then .*

$$\bigcup\big\{F \upharpoonright\!\!\upharpoonright P^c_\xi \ \big|_{p,F}\ p \in P^c_\xi,\ F\ p\text{-rud rec}\big\} \subseteq P^c_{\xi+\xi}.$$

That implies a form of replacement for provident sets:

5·46 COROLLARY *If A is provident, $c \in A$, $\xi \in A$, then there is a $d \in A$ such that*

$$\bigcup\big\{F \upharpoonright\!\!\upharpoonright P^c_\xi \ \big|_{p,F}\ p \in P^c_\xi,\ F\ p\text{-rud rec}\big\} \subseteq d.$$

Comparing two definitions of the constructible hierarchy

We may now compare our definition of $L(c)$ with the traditional one: our version of the latter would be:

5·47 DEFINITION Let c be a transitive set; define

$$T_0(c) = c;\ T_{\nu+1}(c) = \mathbb{T}(T_\nu(c));\ T_\lambda(c) = \bigcup_{\nu,\lambda} T_\nu(c);\ L(c) = \bigcup_{\nu} T_\nu(c).$$

5·48 PROPOSITION *Let c be transitive. For any limit ordinal λ, $T_\lambda(c) \subseteq P^c_{\varrho(c)+\lambda}$; if θ is indecomposable and strictly greater than the rank of c, then $T_\theta(c) \subseteq P^c_\theta$.*

Proof: The sequence $T_\nu(c)$ is given by a c-rud recursion, $c \in P^c_\theta$ and P^c_θ is provident. ⊣ (5·48)

5·49 PROPOSITION *For any limit ordinal λ, $P^c_\lambda \subseteq T_\lambda(c)$.*

Proof: We define f to be a *P-attempt* if f is a function, with domain some ordinal, satisfying these conditions for ordinals in its domain:

$$\begin{aligned} f(0) &= \varnothing \\ f(\nu+1) &= \mathbb{T}(f(\nu)) \cup \{c \cap f(\nu)\} \cup (c \cap \{x \mid x \subseteq f(\nu)\}) \\ f(\lambda) &= \bigcup_{\nu<\lambda} f(\nu)) \end{aligned}$$

All that is $\Delta_0(f, c)$. Now $P^c_\omega = V_\omega \subseteq T_\omega(c)$; so in a suggestive notation, $P \upharpoonright \omega \subseteq T_\omega(c)$.

Suppose that $P \upharpoonright \lambda \subseteq T_\lambda(c)$. Then $P \restriction \lambda$ is a Δ_0 subset of $T_\lambda(c) \times \lambda$ which is a member of $T_{\lambda+\omega}(c)$, and therefore is itself in $T_{\lambda+\omega}(c)$— a couple is a member of $P \restriction \lambda$ if there is an P-attempt in $T_\lambda(c)$ that says so, and $T_\lambda(c)$ is a member of $T_{\lambda+\omega}(c)$.

Now progress as before using the appropriate H_G. ⊣ (5·49)

5·50 COROLLARY *Let $\nu \geqslant \theta > \varrho(c)$, where θ is indecomposable. Then $T_\nu(c) = P^c_\nu$.*

Proof: $T_\theta(c) = P^c_\theta$, by the above observations; thereafter an induction on ν will proceed smoothly at limit stages, and at successor stages we observe that $T_{\nu+1}(c) = \mathbb{T}(T_\nu(c)) = \mathbb{T}(P^c_\nu) = P^c_{\nu+1}$, since $c \subseteq P^c_\theta$ and $c \in P^c_\theta$. ⊣ (5·50)

5·51 REMARK As $c \in T_\theta(c)$, the condition on the rank of c is essential if $T_\theta(c)$ is to be provident.

Miscellaneous results about provident sets

5·52 PROPOSITION *If θ is an indecomposable ordinal and C is a set of transitive sets such that any two members of C are members of a third, then $B =_{\rm df} \bigcup_{c\in C} P^c_\theta$ is provident.*

More generally, the union of a directed system of provident sets is provident.

Proof : Given a parameter p in B and an argument x in B, choose $c \in C$ with both p and x in P^c_θ. We know that P^c_θ is provident, and so if F is p-rud rec, $F(x)$ is in P^c_θ and therefore in B. ⊣ (5·52)

5·53 PROPOSITION *Let A be a provident set, and write $\theta(A)$ for the least ordinal not in A.*

(5·53·0) *A is rud closed;*

(5·53·1) *A contains the rank $\varrho(x)$ of each member x of A;*

(5·53·2) *A contains the transitive closure of each of its members;*

(5·53·3) *$\theta(A)$ is indecomposable;*

(5·53·4) *$\theta(A) = \varrho(A)$;*

(5·53·5) $A = \bigcup_{a\in A} T_{\theta(A)}(a) = \bigcup\{P^d_{\theta(A)} \mid_d \bigcup d \subseteq d \in A\}$

Proof :

(5·53·0) : since each rud function is pure rud rec;

(5·53·1) : since the rank function ϱ is pure rud rec;

(5·53·2) : since tcl is pure rud rec;

(5·53·3) : for each ordinal ζ the function $\nu \mapsto \zeta + \nu$ is ζ-rud rec; hence if ζ and ν are both less than $\theta(A)$, their sum will be too.

(5·53·4) : follows from the first two parts and the transitivity of A.

(5·53·5) : Let $a \in A$; let $d = \mathrm{tcl}(\{a\})$. Then $d \in A$ and $\varrho(d) = \varrho(a)$. Consider the d-rudimentary recursion

$$D(x) = d \cup \bigcup_{y\in x} \mathbb{T}(D(y)).$$

A, being provident, is closed under D. But that recursion is a familiar one in disguise: consider the recursion on ordinals given by $T_0(d) = d$; $T_{\nu+1}(d) = \mathbb{T}(T_\nu(d))$; $T_\lambda(d) = \bigcup_{\nu<\lambda} T_\nu(d)$. It is easily proved by induction on $\varrho(x)$ that for all x, $D(x) = T_{\varrho(x)}(d)$. Hence for $\xi < \theta(A)$, $a \in T_\xi(d) \in A$.

For the second equality use Corollary 5·50. ⊣ (5·53)

5·54 PROPOSITION *Let θ be an indecomposable ordinal, and let $(Q_\nu)_{\nu\leqslant\theta}$ be a θ-progress with $Q_\theta = \bigcup_{\nu<\theta} Q_\nu$. Then Q_θ is provident.*

Proof : i. If $\nu < \theta$, $Q_\nu \in \mathbb{T}(Q_\nu) \subseteq Q_{\nu+1}$ so $Q_\nu \in Q_\theta$, so in Q_θ, everything is a member of a transitive set.

ii. Q_θ is rud closed.

iii. Let $c \in Q_\theta$ be transitive. We show that $P^c_\theta \subseteq Q_\theta$.

iv. By i – iii, Q_θ is the union of a directed family of provident sets, and is therefore provident.

It remains to prove (iii). Let $c \in Q_\eta \in Q_\theta$. $P^c_0 = \varnothing$. We prove that for each non-successor $\lambda < \theta$ and each $k \in \omega$

$\Phi(\lambda)$: $P^c \upharpoonright \lambda \subseteq Q_{\eta+\lambda}$

$\Psi(\lambda, k)$: $P^c \upharpoonright \lambda + k \in Q_{\eta+\lambda+\omega}$.

$\Phi(0)$ is trivial.

Proof that (for λ a non-successor) $\Phi(\lambda) \implies \forall k{:}{\in}\,\omega\ \Psi(\lambda, k)$: $P^c \upharpoonright \lambda$ is uniformly definable from c over $Q_{\eta+\lambda}$ which is in $Q_{\eta+\lambda+1}$, as is c; let c_P be the rudimentary constant for the relevant Δ_0 separator. Then $P^c \upharpoonright \lambda \in Q_{\eta+\lambda+1+c_P}$. There is a rud function H such that for every k, $P^c \upharpoonright (\lambda + k + 1) = H(c, P^c \upharpoonright (\lambda + k))$. Hence if c_H is the corresponding rudimentary constant, then for each $k > 0$, $P^c \upharpoonright \lambda + k \in Q_{\eta+\lambda+1+c_P+k\cdot c_H}$.

That (for λ a nonsuccessor) $\forall k{:}{\in}\,\omega\ \Psi(\lambda, k) \implies \Phi(\lambda + \omega)$ is evident, as is the fact that if λ is a limit of smaller limit ordinals for each of which Φ holds then $\Phi(\lambda)$ holds. ⊣ (5·54)

5·55 DEFINITION Let θ be a limit ordinal. Call two strict continuous θ-progresses $(P_\nu)_{\nu\leqslant\theta}$, $(Q_\nu)_{\nu\leqslant\theta}$ *linked* if for each $\nu \leqslant \theta$, $P_\nu \subseteq Q_\nu$.

5·56 PROPOSITION *Let F be p-rud rec, where $p \in Q_0$; let its constants be s_F and g_F. Suppose that $F \upharpoonright P_0 \subseteq Q_0$. Then for each non-successor λ and $k \in \omega$, $F \restriction P_{\lambda+k} \in Q_{\lambda+s_F+kg_F}$, and $F \upharpoonright P_{\lambda+\omega} \subseteq Q_{\lambda+\omega}$.*

Proof : Assume that for a given non-successor ordinal λ,

$$\begin{aligned} F \upharpoonright P_\lambda &\subseteq Q_\lambda, && \text{then}\\ F \restriction P_\lambda &\in Q_{\lambda+s_F} && s_F \text{ the separational delay}\\ F \restriction P_{\lambda+k} &\in Q_{+\lambda+s_F+g_F k} && g_F \text{ the generational delay; so}\\ F \upharpoonright P_{\lambda+\omega} &\subseteq Q_{\lambda+\omega}. \end{aligned}$$

⊣ (5·56)

5·57 COROLLARY *For no p is the map $\alpha \mapsto \alpha + \omega$ p-rud rec.*

Proof : If F is p-rud rec and θ is indecomposable and strictly greater than $\varrho(p)$, then $\varrho(F(\theta)) < \theta + \omega$. ⊣ (5·57)

A more general statement is true:

5·58 PROPOSITION *let Q_ν be a progress; let e, a transitive set, be in Q_0 and let $P^e_\xi \in Q_0$. Then for each limit ordinal λ, $P^e \upharpoonright \xi + \lambda \subseteq \bigcup_{\nu<\lambda} Q_\nu$.*

Proof by induction on λ: each $\bigcup_{\nu<\lambda} Q_\nu$ is rud closed, and at each successor step we apply a rud function. To continue the induction after reaching a limit ordinal, we remark that if $P^e \upharpoonright \xi + \lambda \subseteq \bigcup_{\nu<\lambda} Q_\nu$, then $P^e \upharpoonright \xi + \lambda \subseteq Q_\lambda \in Q_{\lambda+1}$; hence the sequence so far, $P^e \restriction \lambda$ is definable from P^e_ξ and will therefore be in the rud closed set $\bigcup_{\nu<\lambda+\omega} Q_\nu$. (or the sequence of pairs ?). ⊣ (5·58)

A remark on Type III recursions

5·59 PROPOSITION *A transitive set A is provident iff it contains the graphs of the relevant restrictions of Type III recursions.*

Proof : Let A be provident, and containing an infinite set. Let $F(v, x) = G(v, F \restriction (\{v\} \times x))$, where G is rud. We have seen that for each v, $x \mapsto F(v, x)$ is v-rud-rec. Let d and c be transitive members of A with c of limit rank. Then the proposition gives us a $q \in A$, rud closed and transitive and (necessarily) of limit rank, such that for each $v \in d$ something happens: but that means that all the values taken by F on $d \times c$ are in q, as are the restrictions necessary. So the graph will be

$$(q \times (d \times c)) \cap \{(y, (v, x)) \mid \exists f :\in q \ \ f \text{ is an } F \restriction \{v\} \times u \text{ attempt and} f(v, x) = y\}$$

and thus a set of A, being the result of applying a Δ_0 separator to a set.

Conversely, if $p \in A$ and F_1 satisfies the Type II recursion $F_1(x) = G_1(p, F_1 \restriction x)$, consider the Type III recursion given by $F(v, x) = G(v, F \restriction (\{v\} \times x)$, where G is G_1 appropriately rewritten. ⊣ (5·59)

Provident closures

6·0 THEOREM *Suppose that M is a non-empty set. Let θ be the least indecomposable ordinal not less than $\varrho(M)$. Set*

$$Prov(M) =_{\rm df} \bigcup\{P_\theta^{{\rm tcl}(s)} \mid s \in \mathcal{S}(M)\}.$$

Then $Prov(M)$ is provident and includes M, and if P is any other such, $Prov(N) \subseteq P$.

Proof : Suppose first that P is provident and $M \subseteq P$. Then $\mathcal{S}(M) \subseteq P$; $\theta \leqslant On \cap P$; for each $s \in \mathcal{S}(M)$, ${\rm tcl}(s) \in P$, and for $\nu < \theta$, $P_\nu^{{\rm tcl}(s)} \in P$, and so $Prov(M) \subseteq P$.

Write M_s for $P_\theta^{{\rm tcl}(s)}$; each M_s is provident by the results of §5, and so we have only to remark that if s and t are in $\mathcal{S}(M)$, so is $u = s \cup t$, and $M_s \cup M_t \subseteq M_u$; so that $Prov(M)$ is closed under pairing. ⊣ (6·0)

The theory PROV

Theorem 6·0 implies that there is a finitely axiomatisable set theory which we call PROV of which the transitive models are the provident sets.

Let PROV be the following axioms

(6·0·0) extensionality

(6·0·1) the ten axioms of GJ_0, as given in [M5]

$\varnothing \in V$	$\bigcup x \in V$	$a \cap \{(x,y)_2 \mid x \in y\} \in V$
$\{x, y\} \in V$	${\rm Dom}\,(x) \in V$	$\{(y,x,z)_3 \mid (x,y,z)_3 \in b\} \in V$
$x \smallsetminus y \in V$	$x \times y \in V$	$\{(y,z,x)_3 \mid (x,y,z)_3 \in c\} \in V$

(R_8) $$\{x``\{w\} \mid w \in y\} \in V$$

(6·0·2) `each set is in the domain of an attempt at the rank function`;

(which implies both TCo and set foundation)

(6·0·3) `any two ordinals are in the domain of an attempt at ordinal addition`

(6·0·4) `for each transitive` $\mathfrak{c}$ `each ordinal is in the domain of an attempt at the sequence` $\langle \mathsf{P}_\nu^{\mathfrak{c}} \mid \nu \mathrel{\epsilon} ON\rangle$;

We write PROVI for PROV + $\omega \in V$.

That will suffice to prove that the transitive models of PROV are the provident sets; the reasoning in this paper has been mainly semantic, but experience of the weak systems in [M3] suggest that if one wished to use PROV for syntactical reasoning, it would be desirable to enhance it by adding the axiom of infinity and the scheme of Π_1 foundation. With a little extra work one could show that that too is finitely axiomatisable, by using the predicate $\overset{0}{\models}$.

7: Propagation through levels of the Gödel and Jensen hierarchies

Provident levels of the Jensen hierarchy

The first two statements are parameter-free versions of results already proved.

7·0 LEMMA *Let F be pure rud recursive, given by G. Then "f is an F-attempt" is a Δ_0 predicate of f.*

Proof : Here the formula required is

$$Fn(f) \ \& \ \bigcup \mathrm{Dom}\,(f) \subseteq \mathrm{Dom}\,(f) \ \& \ \forall x \mathbin{:\in} \mathrm{Dom}\,(f) \ \ f(x) = G(f \restriction x). \qquad \dashv (7\cdot 0)$$

7·1 PROPOSITION *If u is transitive and $\varnothing$-provident then so is $rud(u)$.*

Proof : We take $P_n = \mathbb{T}^n(u)$, and $P_\omega = \bigcup_n P_n$. $\langle P_\nu \mid \nu \leqslant \omega\rangle$ is then a strict continuous ω-progress, so we may apply Proposition 5·44 with $p = \varnothing$. ⊣ (7·1)

7·2 COROLLARY *Each non-empty J_ν is $\varnothing$-provident,*

Proof : $J_1 = \mathbf{HF}$; $J_{\nu+1} = \mathrm{rud}(J_\nu)$; the induction at limit stages is trivial. ⊣ (7·2)

The following is a corollary of Theorem 5·47.

7·3 THEOREM *J_ν is provident iff $\omega\nu$ is indecomposable.*

More generally, if c is a transitive set, $J_\nu(c)$ will be provident iff $\omega\nu$ is indecomposable and strictly greater than the rank of c.

7·4 REMARK We need $\omega\nu$ to exceed the rank of c, as provident sets contain the ranks of their members.

7·5 REMARK So although for a given p in L we must go to the first indecomposable ordinal above the moment of construction of p to find a J_ν which is p-provident, every subsequent J_ξ will also be p-provident.

7·6 PROPOSITION *J_ω is provident. The next one will be J_{ω^2}.*

Provident levels of the L hierarchy

7·7 LEMMA *If u is transitive and $u \in L_{\nu+1}$ then $\mathbb{T}(L_\nu) \subseteq L_{\nu+1}$ and $\mathbb{T}(u) \in L_{\nu+2}$.*

Proof : As $u \subseteq L_\nu$, $\mathbb{T}(u)$ is a definable collection of definable subsets of u. u itself is a definable subset of L_ν, so each member of $\mathbb{T}(u)$ is in $L_{\nu+1}$ and $\mathbb{T}(u)$ will be in $L_{\nu+2}$. ⊣ (7·7)

7·8 PROPOSITION *Each L_λ is $\varnothing$-provident for limit λ.*

Proof : $L_\omega = \mathbf{HF}$ which is provident; given this good start, the corollary is easily proved by induction on λ, using the Proposition to advance from λ to $\lambda + \omega$ by taking $P_k = L_{\lambda+k}$, since that is a strict continuous progress. At limit limit ordinals,the induction is trivially maintained. ⊣ (7·8)

7·9 PROPOSITION *L_λ is provident iff λ is indecomposable.*

$\mathcal{S}$-logic in provident sets

7·10 PROPOSITION *Let A be a provident set; let $a \in A$. Then $\mathcal{S}(a) \in A$.*

Proof : If $\omega \notin A$, $A = \mathbf{HF}$; and $a \in \mathbf{HF} \implies \mathcal{S}(a) = \mathcal{P}(a) \in \mathbf{HF}$.

Suppose therefore that $\omega \in a$. Define the following recursion on $\omega + 1$:

$$S(a,0) = \{\varnothing\}; \quad S(a,n+1) = \{\{x\} \cup y \mid x \in a \ \& \ y \in S(a,n)\} \cup S(a,n); \quad S(a,\omega) = \bigcup_{n<\omega} S(a,n).$$

Then $S(a,\omega) = \mathcal{S}(a)$. ⊣ (7·10)

7·11 REMARK If A is provident and $\omega \in A$, then $V_\omega \in A$: for $V_\omega = P^\varnothing_\omega$.

8: Rudimentary recursion from predicates.

Let B be a unary predicate. The class of functions *rudimentary in* B is that obtained by adding to the generators of $\mathcal{R}$ the function $x \mapsto x \cap B$.

We call a function *rud rec in* B if it is of the form

$$F(x) = G(F \restriction x)$$

where G is rud in B; similarly we call it *p-rud-rec in* B if of the form

$$F(x) = G(p, F \restriction x)$$

where G is again rud in B.

That will give us a notion of B-provident: namely a set A is B-provident if whenever p and x are in A and F is p-rud rec in B, then $F(x) \in A$.

We should generalise the definition of $\mathbb{T}$ to $\mathbb{T}^B$: the simplest definition seems to be

$$\mathbb{T}^B(u) = \mathbb{T}(u) \cup \{u \cap B\}.$$

One could consider other definitions, such as to take $\mathbb{T}^B(u)$ to be $\mathbb{T}(u) \cup \{x \cap B \mid x \in \mathbb{T}(u)\}$, which would have the property that as soon as x becomes available so does $x \cap B$. But we shall prefer simplicity to speed.

Then we would wish to define a progress $P_\nu^{c;B}$ where c is a transitive set. Again the simplest would be to replace $\mathbb{T}$ by $\mathbb{T}^B$, and to do nothing else; thus we should have this definition:

8·0 Definition

$$c_0 = \varnothing \qquad c_{\nu+1} = c \cap \{x \mid x \subseteq c_\nu\} \qquad c_\lambda = \bigcup_{\nu<\lambda} c_\nu$$

$$P_0^{c;B} = \varnothing \qquad P_{\nu+1}^{c;B} = \mathbb{T}(P_\nu^c) \cup \{P_\nu^{c;B} \cap B\} \cup \{c_\nu\} \cup c_{\nu+1} \qquad P_\lambda^{c;B} = \bigcup_{\nu<\lambda} P_\nu^{c;B}$$

The Propagation Lemma

8·1 Lemma *Let G be a binary rudimentary-in-B function. Then there is a ternary rudimentary-in-B function H_G, obtainable uniformly from G, such that for any set p, if F be the p-rud-rec-in-B function, given by the recursion $F(x) = G(p, F \restriction x)$, and if P^+ and P are transitive sets with $P \subseteq P^+ \subseteq \mathcal{P}(P)$, then*

$$F \restriction P^+ = H_G(p, F \restriction P, P^+).$$

Proof : If $x \in P^+$, then $x \subseteq P$, so $F \restriction x = (F \restriction P) \restriction x$ so $F(x) = G(p, (F \restriction P) \restriction x)$. Hence

$$F \restriction P^+ = \{(G(p, (F \restriction P) \restriction x), x)_2 \mid_x x \in P^+\}$$

We take $H_G(p, f, q) = \{(G(p, f \restriction x), x)_2 \mid_x x \in q\}$. ⊣ (8·1)

The progression lemma will apply: at limit stages we shall get a set which is rud-in-B closed, so rud closed and B-amenable.

A remark on amenability

8·2 DEFINITION Let A be a set or class, and let C be a transitive class. We define a hierarchy $P_\nu^{C;A}$ thus:

$$P_0^{C;A} = \varnothing; \quad P_{\nu+1}^{C;A} = \mathbb{T}(P_\nu^{C;A}) \cup \{P_\nu^{C;A} \cap A\} \cup \{C \cap P_\nu^{C;A}\} \cup (C \cap \{x \mid x \subseteq P_\nu^{C;A}\}); \quad P_\lambda^{C;A} = \bigcup_{\nu<\lambda} P_\nu^{C;A}.$$

REMARK If A and C are sets, that can be regarded as a rud recursion with them as parameters. If they are proper classes, we have to see that as an A-C-rud recursion with them as predicates.

PROBLEM Let θ be indecomposable, and C transitive of rank θ. When is the rud fattening of C going to be provident ? I suspect that P_θ^C will be the provident closure (or fattening) of C. Cf Brussels.

8·3 PROPOSITION *Let θ be indecomposable, and let $(Q_\nu)_{\nu\leqslant\theta}$ be a θ-progress that is continuous at θ. Let A be amenable to Q_θ, so that $\forall\nu :<\theta\ (A \cap Q_\nu \in Q_\theta)$. Let C be a transitive subset of Q_θ. Then $P_\theta^{C;A}$ is a subset of Q_θ and is A-provident in the predicate sense.*

8·4 REMARK To generalise to C of rank exceeding that of Q_θ (assuming $Q_0 \neq \varnothing$), we should require $C_\theta \subseteq Q_\theta$.

Dynamic predicates

In fact in [M4] we shall have a use for a progress $P^{c;D}$ where the relation D is itself being defined as the progress advances.

Suppose that A is provident and that $D \subseteq A$ is an amenable relation, defined by a p-rud recursion, using the rud functions G_D and H_D. Let c be a transitive set of which p is a member. We define by a simultaneous p-rudimentary recursion sequences $(c_\nu)_\nu$, $(P_\nu^{c;D})_\nu$, $(D_\nu)_\nu$ thus:

8·5 DEFINITION

$$\begin{array}{lll} c_0 = \varnothing & c_{\nu+1} = c \cap \{x \mid x \subseteq c_\nu\} & c_\lambda = \bigcup_{\nu<\lambda} c_\nu \\ P_0^{c;D} = \varnothing & P_{\nu+1}^{c;D} = \mathbb{T}(P_\nu^{c;D}) \cup \{c_\nu\} \cup c_{\nu+1} \cup \{P_\nu^{c;D} \cap D_\nu\} & P_\lambda^{c;D} = \bigcup_{\nu<\lambda} P_\nu^{c;D} \\ D_0 = \varnothing & D_{\nu+1} = H_D(p, D_\nu, P_{\nu+1}^{c;D}) & D_\lambda = \bigcup_{\nu<\lambda} D_\nu \end{array}$$

9: Functions defined by more liberal recursions

We may ask if provident sets are closed under more functions than those we have discussed: for example, if G is u.a.d., and F is defined by $F(x) = G(F \restriction x)$, will a provident set be closed under F ?

The matter appears to be delicate, for provident sets need not be closed under ordinal multiplication, which is defined by the recursion

$$\alpha \cdot 0 = 0; \quad \alpha \cdot (\beta + 1) = \alpha \cdot \beta + \beta; \quad \alpha \cdot \lambda = \sup_{\nu < \lambda} \alpha \cdot \nu.$$

But notice that in the recursive definition of $\alpha + \beta$, the recursion, which is rudimentary, is on the second variable, and α is a parameter not involved in the recursion; whereas in the definition of $\alpha \cdot (\beta + 1)$, the value computed previously for $\alpha \cdot \beta$ is the value given to the first variable when ordinal addition is called; so that both variables of $\lambda\alpha\beta.\alpha + \beta$ are involved in the recursive definition of $\lambda\alpha\beta.\alpha \cdot \beta$.

What we find, though, is that if such "crossing" of variables is avoided, then provident sets will be closed under functions generated by recursions using previously obtained functions.

9·0 Definition We say that a unary class function F is d.f.d. (meaning definable and of finite delay) if there is a parameter p_F and a function $\ell_F : \omega \longrightarrow \omega$ such that for each canonical progress P^c_ν, where c is a transitive set of which p is a member, each non-successor ordinal λ and each $k \in \omega$,

$$F \restriction P^c_{\lambda+k} \subseteq P^c_{\varrho(p)+1+\lambda+\ell_F(k)}.$$

Thus we have shown that every rudimentarily recursive function is d.f.d., with affine delay.

In the following remarks, we assume without proof that the notion of definability just given is adequate for its use. Our aim here is simply to explore the rate of growth of various definitions.

9·1 Proposition *Suppose that $F_3 = F_2 \circ F_1$, where for $i = 1, 2$ F_i is d.f.d with parameter p_i and delay function ℓ_i. Then F_3 is d.f.d.*

Proof : as in the proof of Theorem 5·41, we may argue that:

$$\begin{aligned} & F_1 \restriction P^c_{\lambda+k} \in P^c_{\varrho(p_1)+\lambda+\ell_1(k)} \\ \text{and} \quad & F_2 \restriction P^c_{\varrho(p_1)+\lambda+\ell_1(k)} \in P^c_{\varrho(p_2)+\varrho(p_1)+\lambda+\ell_2(\ell_1(k))}; \\ \text{but} \quad & F_3 \restriction P^c_{\lambda+k} = \big(F_2 \restriction P^c_{\varrho(p_1)+\lambda+\ell_1(k)}\big) \circ \big(F_1 \restriction P^c_{\lambda+k}\big) \\ \text{so} \quad & F_3 \restriction P^c_{\lambda+k} \in P^c_{\varrho(p_2)+\varrho(p_1)+\lambda+\ell_2(\ell_1(k))+c_\circ}; \\ \text{thus we shall have} \quad & F_3 \restriction P^c_{\lambda+k} \in P^c_{\varrho(p_3)+\lambda+\ell_3(k)} \end{aligned}$$

as required, if we set $\ell_3(k) = \ell_2(\ell_1(k)) + c_\circ$, and p_3 to be some set of rank at least $\varrho(p_2) + \varrho(p_1)$ that includes $\{p_1, p_2\}$. ⊣ (9·1)

9·2 Remark If F_2 is rudimentary, $\ell_3(k) = \ell_1(k) + c_\circ + c_{F_2}$; if F_1 is rudimentary, $\ell_3(k) = \ell_2(k + c_{F_1}) + c_\circ$.

9·3 Remark If $\ell_1(k)$ is exponential and $\ell_2(k)$ is affine, or if $\ell_2(k)$ is exponential and $\ell_1(k)$ is affine, then ℓ_3 will be exponential.

9·4 Remark If $\ell_1(k)$ is $O(g_1^k)$ and $\ell_2(k)$ is $O(g_2^k)$, ℓ_3 will be $O(g_2^{g_1^k})$.

9·5 PROPOSITION *Suppose that G is of d.f.d. with parameter p and delay function ℓ. Suppose that F is defined by $F(x) = G(F\restriction x)$. Then F is likewise d.f.d.*

Proof : We need some definability condition to say that $F\restriction P^c_\lambda$ is definable over it and therefore will be in $P^c_{\lambda+s_F}$. Thereafter

$$F\restriction P_{\lambda+k+1} = H_G(p, F\restriction P_{\lambda+k}, P_{\lambda+k+1})$$

where we define H_G as before:

$$H_G(p, f, q) \equiv \{(G(p, f\restriction x), x)_2 \mid_x x \in q\}.$$

H_G is of the form $K_1 \circ (G \circ K_2)$ where K_1 and K_2 are rudimentary; so the delays of G and H_G are of the same order. In particular, if G is u.a.d. so is H_G.

Now assuming that $\ell_F(k) \geqslant k+1$, both arguments of H_G are in $P_{\lambda+\ell_F(k)}$, and $H_G \restriction P_{\lambda+\ell_F(k)} \in P_{\lambda+\ell_H(\ell_F(k))}$, so we obtain the recursive estimate

$$\ell_F(k+1) = \ell_H(\ell_F(k)). \qquad \dashv (9{\cdot}5)$$

9·6 REMARK Thus if G is rud and therefore H_G is rud, $\ell_H(k) = k + c_H$, (roughly): so

$$\ell_F(0) = s_F; \quad \ell_F(1) = s_F + c_H; \quad \ell_F(2) = s_F + c_H + c_H; \quad \dots \quad \ell_F(m) = s_F + m\cdot c_H,$$

giving the uniform affine delay that we have already established for rud rec functions.

9·7 REMARK If H is u.a.d., there are delays s and g such that $\ell_H(k) = s + kg$, and we obtain

$$\ell_F(0) = s_F; \quad \ell_F(1) = s + s_F g; \quad \ell_F(2) = s + (s + s_F g)g; \quad \ell_F(3) = s(1 + g + g^2) + s_F g^3;$$

and in general

$$\ell_F(m) = s(\sum_{i=0}^{m-1} g^i) + s_F g^m$$

which is $O(g^m)$, so that it is reasonable to describe F as being of uniform exponential delay.

9·8 REMARK If G and therefore H is of exponential delay, the delay for F will grow even faster: if, to simplify the picture, we suppose that $\ell_H(m) = g^m$ and $\ell_F(0) = s$, then

$$\ell_F(1) = g^s; \quad \ell_F(2) = g^{g^s}; \quad \ell_F(3) = g^{g^{g^s}};$$

and so on. $\dashv$ (9·8)

Limit provident sets

9·9 REMARK The function of ordinals defined by $F(0) = \eta$; $F(\nu+1) = \eta + F(\nu); F(\lambda) = \sup_{\nu<\lambda} F(\nu)$ is of the form $F(\nu) = E(\eta, F \restriction \nu)$ where E is (related to) the η-rud rec function $\alpha \mapsto \eta + \alpha$, and has the property that for all $\nu \geqslant \eta$, $F(\nu) = \eta \cdot \omega$.

Thus the rank of a provident set A containing ω and closed under all F's of the above type, must be ω^λ for some limit ordinal λ.

We wish here to prove the converse.

9·10 THEOREM *Suppose that A is provident. Suppose that F is a class function satisfying the recursion $F(x) = E(F \restriction x)$ where for some $p \in A$, E itself is p-rud rec, given by $E(x) = G(p, E \restriction x)$, where G is rudimentary. Suppose further that $\eta =_{\rm df} \varrho(p) \cdot \omega < \theta =_{\rm df} On \cap A$. Then A is closed under F.*

9·11 LEMMA *There is a function $\ell_F : \omega \longrightarrow \omega$ such that for every transitive set e with $p \in e$, every nonsuccessor ordinal λ and every natural number k, $F \restriction P^e_\lambda \subseteq P^e_{\varrho(p)\cdot\omega+\lambda}$ and $F \restriction P^e_{\lambda+k} \in P^e_{\varrho(p)\cdot\omega+\lambda+\ell_F(k)}$.*

Proof : As usual assume that for some λ, $F \restriction P^e_\lambda \subseteq P^e_{\varrho(p)\cdot\omega+\lambda}$. $F \restriction P^e_\lambda$ is uniformly definable from p over $P^e_{\varrho(p)\cdot\omega+\lambda}$ and therefore will be in $P^e_{\varrho(p)\cdot\omega+\lambda+c}$ for some finite c which we call $\ell_F(0)$.

Thereafter

$$F \restriction P_{\lambda+k+1} = H_E(F \restriction P_{\lambda+k}, P_{\lambda+k+1})$$

where we define H_E as before:

$$H_E(f,q) \equiv \{(E(f \restriction x), x)_2 \mid_x x \in q\}.$$

H_E is of the form $K_1 \circ (E \circ K_2)$ where K_1 and K_2 are rudimentary; so the delays of E and H_E are of the same order. In particular, if E is u.a.d. so is H_E.

Now assuming that $\ell_F(k) \geqslant k+1$, both arguments of H_E are in $P^e_{\varrho(p)\cdot\omega+\lambda+\ell_F(k)}$; since H_E is u.a.d,

$$H_E \restriction P^e_{\varrho(p)\cdot\omega+\lambda+\ell_F(k)} \in P^e_{\varrho(p)+1+\varrho(p)\cdot\omega+\lambda+\ell_{H_E}(\ell_F(k))};$$

but $\varrho(p) + 1 + \varrho(p) \cdot \omega = \varrho(p) \cdot \omega$.

We may therefore define ℓ_F recursively by

$$\ell_F(k+1) = \ell_{H_E}(\ell_F(k)). \qquad \dashv (9{\cdot}11)$$

The proof of the Theorem is now immediate. ⊣ (9·10)

9·12 COROLLARY *If A is provident and $On \cap A = \omega^\lambda$ for some limit ordinal λ, then A is closed under all functions defined by p-rud-rec recursion with p in A.*

9·13 REMARK Thus it appears that whereas for p-rud-rec functions, a good start is some $F \restriction P^c_{\varrho(p)+1}$, for functions defined by recursion using a p-rud-rec function, a good start is some $F \restriction P^c_{\varrho(p)\cdot\omega}$. This phenomenon might continue to rise with the hierarchy that is emerging.

10: Models of stunted growth

We have mentioned "Model $\mathbf{M}_{13,\lambda}$" studied in *Weak Systems*, which is supertransitive and a proper class but which contains only the ordinals $< \lambda$; so in this model rank is stunted.

10·0 ASIDE Consider this model in the special case $\lambda = \omega$; ω is not a member of $\mathbf{M}_{13,\omega}$, which is otherwise a model of Z, save for the axiom of infinity in its customary form. But that axiom is not used in defining the finite basis of rudimentary functions; so $\mathbf{M}_{13,\omega}$ is rud closed; and therefore ω is not of the form $F(x)$ for any rud function F and $x \in \mathbf{M}_{13,\omega}$.

That argument proves another variant of Gandy's Theorem 2.1.3. Note that in Model $\mathbf{M}_{13,\lambda}$, TCo holds; by supertransitivity, the actual transitive closure of each member of the model is a member of the model.

We may generalise the idea behind model $\mathbf{M}_{13}$ thus:

10·1 DEFINITION Suppose that $F : On \longrightarrow V$ is a function such that for each ζ, $F(\zeta) \in F(\zeta+1) \subseteq \mathcal{P}(F(\zeta))$, that $F(0) = \varnothing$ and that at a limit stage η, $F(\eta) = \bigcup_{\nu<\eta} F(\nu)$; so that each $F(\zeta)$ is transitive. For limit λ, set

$$A_{F,\lambda} =_{\rm df} \big\{u \mid \bigcup u \subseteq u \;\&\; \sup\{\xi \in u \cap On \mid F(\xi) \in u\} < \lambda\big\}; \quad M_{F,\lambda} = \bigcup A_{F,\lambda}.$$

10·2 PROPOSITION *$M_{F,\lambda}$ is a supertransitive model of Z for which $F(\xi) \in M_{F,\lambda} \iff \xi < \lambda$. For many F and λ the model $M_{F,\lambda}$ will be a proper class.*

Proof : as in Section 7 of [M2]. The union of two members of $A_{F,\lambda}$ is in $A_{F,\lambda}$, and if $u \in A_{F,\lambda}$, so is $\mathcal{P}(u)$; so that $M_{F,\lambda}$ will be a supertransitive model of Z. If F only takes ordinals as values, the argument in M2 shows that $M_{F,\lambda}$ will contain sets of all ranks. Otherwise, if $F(\xi)$ is not an ordinal for $\xi \geqslant \eta$, where $\eta < \lambda$, then $M_{F,\lambda}$ will contain all ordinals.

⊣ (10·2)

10·3 Recall from the Introduction the iteration T_ν of the function $\mathbb{T}$, given by

$$T_0 = \varnothing; \quad T_1 = \{\varnothing\}; \quad T_{\nu+1} = \mathbb{T}(T_\nu); \quad T_\eta = \bigcup_{\nu<\eta} T_\nu \quad \text{for limit } \eta.$$

10·4 DEFINITION For limit λ, set $\mathbf{A}_{17,\lambda} =_{\rm df} \big\{u \mid \bigcup u \subseteq u \;\&\; \sup\{\zeta \mid T_\zeta \in u\} < \lambda\big\}$; $\quad \mathbf{M}_{17,\lambda} =_{\rm df} \bigcup \mathbf{A}_{17,\lambda}$.

10·5 PROPOSITION *$\mathbf{M}_{17,\lambda}$ is a supertransitive proper class, containing all ordinals but the T hierarchy only up to λ but no further. In this model the rud recursion defining rank is total but that defining the growth of the Jensen auxiliary hierarchy stops prematurely.*

10·6 PROPOSITION *There is a supertransitive class model* $\mathbf{M}_{18,\lambda}$ *of* Z *which contains a Cohen generic real* c*, and all constructible sets, but such that neither* $L_{\omega+\omega}(c)$ *nor* $P^c_{\omega+\omega}$ *is in* $\mathbf{M}_{18,\lambda}$.

Proof : This time take $\lambda = \omega+\omega$ and $F(\zeta) = P^c_\zeta$ and $\mathbf{M}_{18,\lambda} = M_{F,\lambda}$. $c \in P^c_\zeta$ whenever $\zeta \geqslant \omega+1$, so that each $L_\eta \in A_{F,\lambda}$ and $L \subseteq M_{F,\lambda}$. ⊣ (10·6)

In the above model the Jensen hierarchy exists for all ordinals, but the same hierarchy relativised to c is defined before but not at level $\omega+\omega$.

10·7 REMARK We have seen that in the model $\mathbf{K}$, which should have been called $\mathbf{M}_{16}$, of section 12 of [M3], the definition of tcl is stunted, and therefore also the definition of rank, for if every set is a member of the domain of some attempt at ϱ, that domain will be a transitive set; so TCo holds, and hence tcl may be recovered using the full strength of the axioms of Z.

$\mathbf{M}_{13}$ is a model of ZC in which rank is stunted but tcl not; $\mathbf{M}_{17}$ is a model of ZC in which the Jensen hierarchy is stunted but tcl and rank not; $\mathbf{M}_{18}$ is a model of ZC in which the relative Gödel and Jensen hierarchies $L_\nu(c)$ and $J_\nu(c)$ are stunted but the hierarchies L_ν and J_ν and tcl and rank are not.

So there is a certain ordering to some rudimentary recursions; but we have seen that if all recursions broadly similar to T_ν are total, and if rank is total, then all rud recursions give total functions, so that in this special sense there is a finite basis to the class of rud recursions.

Failure of Scott's trick in a model of Zermelo

We record here another variant of the above construction.

10·8 DEFINITION Let $A = \leqslant_R$ be a well-ordering, viewed as a binary relation $\leqslant_R$ on the set $\{x \mid \langle x,\, x\rangle \in A\}$. For such A, define $I(A)$ to be the class of well-orderings isomorphic to A, and, in imitation of Scott's celebrated trick for reducing equivalence classes to equivalence sets, let $ST(A)$ be the class $\{B \in I(A) \mid \forall C :\in I(A)\ \varrho(C) \geqslant \varrho(B)\}$, the class of wellorderings of minimal rank isomorphic to A.

The following shows that Z is too weak a set theory for Scott's trick to work.

10·9 THEOREM *Let* $\kappa = \beth_\kappa$ *be a beth fixed point. Let* A_κ *be the epsilon relation restricted to* κ*; thus a well-ordering of length* κ*. There is a supertransitive, proper class, model* $\mathbf{M}_{19}$ *of* Z *containing all ordinals and the well-ordering* A_κ*, in which every set has a rank, but in which* $(ST(A_\kappa))^{\mathbf{M}_{19}}$*, though a definable class of the model, is not a set.*

Proof : Take $F(\nu) = V_\nu$ and $\mathbf{M}_{19} = M_{F,\kappa}$. $V_\nu \in \mathbf{M}_{19} \iff \nu < \kappa$. As κ is a beth fixed point, $V_\kappa = H_\kappa$, so that all well-orderings of length κ in the universe must be of rank at least κ. Thus $A_\kappa \in I(A_\kappa)$. Let B be the well-ordering $\{\langle V_\nu,\, V_\xi\rangle \mid_{\nu,\xi} \nu \leqslant \xi < \kappa\}$, and let B_ξ be the well-ordering $\{\langle b_\nu,\, b_\xi\rangle \mid_{\nu,\xi} \nu \leqslant \xi < \kappa\}$ where for $\zeta \leqslant \xi$, $b_\zeta = V_\zeta$, and for $\xi < \zeta < \kappa$, $b_\zeta = \zeta$.

Then each $B_\xi \in \mathbf{M}_{19}$, being obtained from $V_{\xi+1}$ and A_κ by rudimentary operations. Further each B_ξ is of rank κ, and thus is in $ST(A_\kappa)$, even as defined in $\mathbf{M}_{19}$.

$V_\kappa \notin \mathbf{M}_{19}$, therefore $B \notin \mathbf{M}_{19}$, as V_κ is the union of its field. If $ST(A_\kappa)^{\mathbf{M}_{19}}$ were in the model we could form the set of initial segments of all of its members, using the power set axiom; among those we could pick out all the V_ν's, ($\nu < \kappa$) using the rank function within the model; and then form their union, which would be the forbidden set V_κ. ⊣ (10·9)

11: The truth predicate for $\dot{\Delta}_0$ sentences

In [M3] the theory MW was introduced: it adds to the axioms of DBI the principle that

$$\forall a \forall k {:}{\in}\, \omega \; [a]^k \in V$$

By [M3, Theorem 2.91], MW is a sub-theory of GJI; but in [M3, section 5], a supertransitive class $\mathbf{M}_5$, containing all ordinals, is defined which by [M3, Propositions 5.0 and 5.19 and Remark 5.9] is a model of MW but not of GJ; by [M3, Lemma 5.9], this model omits the set of finite subsets of ω. Further, Model $\mathbf{M}_{6,3}$ of that paper models DBI but not MW, since it omits the set $[\omega]^3$. Thus MW is a theory strictly intermediate between DBI and GJI.

In [M3, section 10, culminating in page 211] it is shown that the truth predicate $\models_u \varphi$ is, provably in MW, Δ_1-definable.

It was promised in [M3] that the proof of Devlin VI.1.14 would be reworked in the present paper. We shall instead present a proof of the following sharper result.

11·0 THEOREM *Truth for $\dot{\Delta}_0$ sentences is uniformly Δ_1 for transitive models of* MW.

Proof: Let M be a transitive model of MW, and φ a $\dot{\Delta}_0$ sentence of $\mathcal{L}_M$. It suffices to find a Σ_1 definition of $\models_M \varphi$, for if a truth predicate for a class of sentences that is closed under negation is Σ_1 it will automatically be Π_1, since $\models \varphi \iff \neg \models \neg\varphi$.

We have a sentence φ; let k be its length, k; let N_φ be the finite set comprising those members of M of which the names occur in φ; let q_φ be the number of occurrences of quantifiers in φ.

Step 1: we rewrite φ by de-nesting restricted quantifiers: for example,

replace $\bigwedge \mathfrak{x} {:}{\in}\, \mathring{a} \bigvee \mathfrak{y} {:}\, \epsilon\, \mathfrak{x}\; \vartheta$ by $\bigwedge \mathfrak{x} {:}{\in}\, \mathring{a} \bigvee \mathfrak{y} {:}\, \epsilon\, \mathring{c}\, [\mathfrak{y} \,\epsilon\, \mathfrak{x} \wedge \vartheta]$, where $c = \bigcup a$.

We thus reach within q_φ steps a formula φ' in which all quantifiers are restricted by *free terms*, each of the form <name of> $\bigcup^m a$, where $a \in N_\varphi$ and $m < q_\varphi$. As the Axiom of Union is among those of MW, each such $\bigcup^m a$ will be in M. Let F_φ be the finite set comprising those members of M of which the names occur in φ'.

A similar process is described in some detail in section 8 of [M3], though there, but not here, the formalism admits limited quantifiers as well as restricted ones.

Step 2: using the usual procedures of predicate logic, we rewrite φ' in prenex form, thus reaching a sentence φ^* in which a string of quantifiers, all restricted by free terms, precedes a quantifier-free formula ϑ.

These two steps may be achieved by primitive recursive processes applied to the formulæ in question.

We must now show that M contains a set which contains all the constants that will occur in substitution instances of subformulæ of φ^*: but such a set will be $P_\varphi =_{\text{df}} F_\varphi \cup \bigcup F_\varphi$

Let S_φ be the class of quantifier-free sentences, of length no longer than k, in which the only names occurring are those of members of P_φ, That, provably in MW, will be a set.

Step 3: we show that truth for members of S_φ is uniformly Σ_1 for transitive models of MW.

Specifically, we show that there is an *evaluation* $g_\varphi : S_\varphi \longrightarrow 2$, that is, a function which obeys the rules for evaluation of Boolean combinations of atomic statements. These rules are:

$$g(\mathring{x} = \mathring{y}) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

$$g(\mathring{x} \,\epsilon\, \mathring{y}) = \begin{cases} 1 & \text{if } x \in y \\ 0 & \text{if } x \notin y \end{cases}$$

$$g(\neg\vartheta) = 1 - g(\vartheta)$$

$$g(\vartheta_1 \wedge \vartheta_2) = \inf\{g(\vartheta_1), g(\vartheta_2)\}$$

$$g(\vartheta_1 \vee \vartheta_2) = \sup\{g(\vartheta_1), g(\vartheta_2)\}$$

and similarly for other propositional connectives if they have also been taken as primitive.

We saw in [M3] that a statement of the form $\vartheta = \vartheta_1 \wedge \vartheta_2$ is not Δ_0 but will be Δ_0 in any sufficiently long attempt at addition. As the sentences to be considered are all of length not exceeding that of φ, a single sufficiently long attempt, α, will exist, and we shall be able to express the above rules for g as a statement that is $\Delta_0(\alpha, g, S_\varphi)$.

Thus the desired Σ_1 truth predicate for sentences ϑ in S_φ will be of the form

$$\exists\alpha,\ \text{a sufficiently long attempt at addition, } \exists g : S_\varphi \longrightarrow 2,\ g \text{ an evaluation, with } g(\vartheta) = 1.$$

Step 4: we show how to reduce the computation of truth of φ^* to that of numerous substitution instances in S_φ.

REMARK This step would be possible even if we had not done Steps One and Two, but would be more complicated to express.

We consider the tree T of all finite sequences $\langle c_0, \dots c_\ell\rangle$ of members of P_φ where $\ell \leqslant n$ and for each i, $c_i \in a_i$. Provably in MW, T is a set. We write $\emptyset$ for the empty sequence.

We define for each $t \in T$ a sentence φ_t by recursion on the length of t.

Let $\varphi_\emptyset = \varphi^*$. If that is quantifier-free, we have nothing more to do; so suppose that it has $n+1$ quantifiers, so that there are $a_0, \dots, a_n$ in M such that $\varphi_\emptyset$ is $\mathcal{Q}_0\mathfrak{x}_0 : \epsilon\, \mathring{a}_0 \mathcal{Q}_1\mathfrak{x}_1 : \epsilon\, \mathring{a}_1 \dots \mathcal{Q}_n\mathfrak{x}_n : \epsilon\, \mathring{a}_n\, \vartheta$ where $\vartheta \in S_\varphi$ but may contain other names besides those shown. n is not greater than k.

Once we have defined φ_t then for $c \in a_{\ell h(t)}$ we define $\varphi_{t^\frown\langle c\rangle}$ to be $\mathrm{Subst}(\varphi_t, \mathfrak{x}_{\ell h(t)}, \mathring{c})$.

Let $T_\varphi = \{\varphi_t \mid t \in T\}$, a tree of sentences.

Let g_φ be the evaluation defined on S_φ in Step 3. Extend it to T_φ by a reverse induction: if $\ell h(t) = n+1$, φ_t will be a member of S_φ, and so $g_\varphi(\varphi_t)$ has been defined in Step 3. If $g_\varphi(\varphi_u)$ has been defined for all $u \in T$ of length $\ell h(t)+1$, then define

$$g_\varphi(\varphi_t) = \begin{cases} \sup\{g(\varphi_{t^\frown\langle c\rangle}) \mid c \in a_{\ell h(t)}\} & \text{if } \mathcal{Q}_{\ell h(t)} \text{ is } \bigvee \\ \inf\{g(\varphi_{t^\frown\langle c\rangle}) \mid c \in a_{\ell h(t)}\} & \text{if } \mathcal{Q}_{\ell h(t)} \text{ is } \bigwedge \end{cases}$$

So $\overset{0}{\models} \varphi \iff g_\varphi(\varphi_\emptyset) = 1$.

We have finally to observe that as M models MW, then for φ a $\dot\Delta_0$ sentence of $\mathcal{L}_M$, all the above sets and functions, in particular P_φ, S_φ, T_φ and g_φ are in M; so the desired Σ_1 formula simply says that there exist sets and functions which obey the rules imposed on them and which lead to the evaluation of φ. ⊣ (11·0)

The same argument with very few changes will give a possibly less laborious proof of the result proved in section 10 of [M3]:

11·1 THEOREM *The truth predicate* $\models_u \varphi$, *for* u *a set and* φ *an arbitrary sentence of* $\mathcal{L}_u$, *is* Δ_1^{MW}.

Suppose for simplicity that φ has no restricted quantifiers, has $n+1$ unrestricted quantifiers and is prenex, thus of the form $\mathcal{Q}_0\mathfrak{x}_0\mathcal{Q}_1\mathfrak{x}_1 \dots \mathcal{Q}_n\mathfrak{x}_n\,\vartheta$ where ϑ is quantifier-free but may contain names of some members of u. n is not greater than k.

We form the set $S_{u,k}$ of quantifier-free sentences of $\mathcal{L}_u$ of length no greater than φ, and find an evaluation g defined on it obeying the above rules.

This time take the tree T to be the set of all finite sequences $\langle c_0, \dots c_\ell\rangle$ where $\ell \leqslant n$ and for each i, $c_i \in u$. As before, we define for each $t \in T$ a sentence φ_t by recursion on the length of t. We set $\varphi_\emptyset = \varphi$; once we have defined φ_t we set, for $c \in u$, $\varphi_{t^\frown\langle c\rangle}$ to be $\mathrm{Subst}(\varphi_t, \mathfrak{x}_{\ell h(t)}, \mathring{c})$.

Let $T_\varphi = \{\varphi_t \mid t \in T\}$. We extend the definition of g to members of T_φ by modifying our previous definition by reverse induction: if $g_\varphi(\varphi_u)$ has been defined for all $u \in T$ of length $\ell h(t)+1$, then define

$$g_\varphi(\varphi_t) = \begin{cases} \sup\{g(\varphi_{t^\frown\langle c\rangle}) \mid c \in u\} & \text{if } \mathcal{Q}_{\ell h(t)} \text{ is } \bigvee \\ \inf\{g(\varphi_{t^\frown\langle c\rangle}) \mid c \in u\} & \text{if } \mathcal{Q}_{\ell h(t)} \text{ is } \bigwedge \end{cases}$$

Then $\models_u \varphi \iff g_\varphi(\varphi_\emptyset) = 1$.

As we are arguing in MW and u is a set, then for φ any sentence of $\mathcal{L}_u$, all the above sets and functions, in particular P_φ, S_φ, T_φ and g_φ are sets; so the desired Σ_1 formula simply says that there exist sets and functions which obey the rules imposed on them and which lead to the evaluation of φ. ⊣ (11·1)

R E F E R E N C E S

[De] K. Devlin, *Constructibility*, Perspectives in Mathematical Logic, *Springer-Verlag, Berlin*, 1984.

[Do] A. J. Dodd, *The Core Model*, London Mathematical Society Lecture Note Series, 61, Cambridge University Press, 1982. MR **84a:**03062.

[G] R. O. Gandy, Set-theoretic functions for elementary syntax, in *Proceedings of Symposia in Pure Mathematics*, **13**, Part II, ed. T. Jech, American Mathematical Society, 1974, 103–126.

[J1] R. B. Jensen, Stufen der konstruktiblen Hierarchie. Habilitationsschrift, Bonn, 1967 (?)

[J2] R. B. Jensen, The fine structure of the constructible hierarchy, with a section by Jack Silver, *Annals of Mathematical Logic,* **4** (1972) 229–308; erratum ibid **4** (1972) 443.

[JK] R. B. Jensen and C. Karp, Primitive recursive set functions, in *Proceedings of Symposia in Pure Mathematics*, **13**, Part I, ed. D. Scott, American Mathematical Society, 1971, 143–176.

[M1] A. R. D. Mathias, Slim models of Zermelo Set Theory, *Journal of Symbolic Logic* **66** (2001) 487–496.

[M2] A. R. D. Mathias, The Strength of Mac Lane Set Theory, *Annals of Pure and Applied Logic,* **110** (2001) 107–234.

[M3] A. R. D. Mathias, Weak systems of Gandy, Jensen and Devlin, in *Set Theory: Centre de Recerca Matemàtica, Barcelona 2003-4* , edited by Joan Bagaria and Stevo Todorcevic, Trends in Mathematics, Birkhäuser Verlag, Basel, 2006, 149–224.

[M4] A. R. D. Mathias, Provident sets and rudimentary set forcing, *in preparation*.

[M5] A. R. D. Mathias, Unordered pairs in the set theory of Bourbaki 1949, *Archiv der Mathematik, to appear*.

[St] Stanley, Lee, review of [De], *Journal of Symbolic Logic* **53**(1987) 864–8.