

## What is new and what not in *The Strength of Mac Lane Set Theory*

Mostowski, in his paper *A formula with no recursively enumerable model*,\* writes:

*“It has been justly observed that many recent papers in the field of symbolic logic do not supply full proofs of the statements they contain. While it would certainly not be reasonable to require from all papers to give exhaustive proofs, it is certainly necessary to publish full proofs from time to time. This line is followed in the present paper.”*

That too became my line with my paper *The Strength of Mac Lane Set Theory*,† which began as a compilation of known results made for my own use, but which gradually, as I grew dissatisfied with the sketchiness of published arguments, acquired longer and more detailed proofs that led in turn to sharper formulations of those known results and thence to new ones.

I shall now briefly run through the paper saying what is new and what not: “new” means that I am not aware of the result being in print elsewhere; it does not mean that no one else could have proved it. The novelty in many cases is in the point of view rather than in the technical requirements of the proof.

*Of the twenty theorems stated explicitly in the Introduction:*

Theorem 1 is new; the proofs of theorems 2,3,4 develop an idea that has been known in some form for some time.

Theorems 5, 6, 7, are new; the proofs use fine-structural arguments of Jensen, as modified by Sy Friedman.

Theorem 8 gives a new, set-theoretical, proof of a result that proof-theorists, but not set-theorists, would regard as standard.

Theorem 9 is new; perhaps the most surprising result in the paper, its discovery is the result of refining an observation of Harvey Friedman, itself a generalisation of a celebrated lemma of Françoise Ville, to the form given in Proposition 6.40.

Theorem 10, with a different proof, was known to Harvey Friedman in 1972.

Theorem 11 is new, and the method of proof is innovative.

Theorems 12 and 13 sharpen an argument of Coret.

Theorems 14 and 15 are new, as are 16 and 17; 18 is a rediscovery of a theorem of Lake; 19 is new, using a lemma of Forster and Kaye; 20 is new.

Proposition Scheme 7.2, essentially a formal version of the statement that  $H_{\beth_\omega}$  is an elementary extension of  $V_{\omega+\omega}$  for stratifiable formulæ, has been cited as the Mathias–Boffa theorem. I was working from a manuscript of Boffa that expounded Coret’s theorem, and I should have to find it again to know where Boffa’s exposition stopped and my improvement began.

*Reviewing the material by section:*

The central idea of section 1 goes back to von Neumann in the ’20s.

Section 2 is a translation into purely set-theoretical terms of an argument known to the category theorists in the 70’s; in Section 3 some of the detail will be new, but quite a lot will have been known to several groups.

Section 4: a treatment of  $L$ ; many details have not previously appeared in print, particularly the approach in pages 141–143; but it is quite probable that Gödel explored all these approaches in unpublished work in the ’30s: see Historical Note 4.21.

Section 5: an adaptation to this context of ideas developed by Jensen and his co-workers with a different context in mind. I fear the reader will find some passages confusing despite the care I had given to the choice of notation, for, maddeningly, the printers have, most carelessly, not observed certain distinctions between fonts which I had requested with a view to clarifying the subtleties of what are perhaps the hardest arguments of the paper.

Section 6: my aim here was to establish the independence of various statements from Mac Lane set theory, and I found what I believe is a new method of doing so by forcing over non-standard models. I make use in this section of results from a paper of Harvey Friedman from the 70s. Friedman’s principal tool is the

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\* *Fundamenta Mathematicæ* **42** (1955) 125–140.

† *Annals of Pure and Applied Logic*, **110** (2001) 107–234.

Barwise compactness theorem, which I avoid, preferring to use the Gandy basis theorem; and his paper is written in a rather different formalism, which I am at pains to relate to my preferred one.

This section contains what is perhaps the most unexpected result of the paper, Theorem 6.47, which shows that adding the axiom of constructibility to a certain weak theory strictly increases its consistency strength; a phenomenon entirely the opposite to what happens in the case of ZF or of the natural subsystems of Z considered in section 5. The discovery of this result was a consequence of my systematically minimising the assumptions required for each proposition in the section.

As in section 5, the printed text is blighted by the printer's inattention to font distinctions; pages 186 and 187 are particularly badly hit.

Section 7: again a careful exposition of known material has led to new results, in this case to the conservative extension result proved as Theorem 14 on page 196.

Section 8: two pieces of "folk-lore" worked out in careful detail as Theorems 8.3 and Theorem 8.31: see the Historical Notes 8.2 and 8.36. Again the printers have failed to distinguish between two fonts, in this case slanted type and italic type, but the harm done is less than in sections 5 and 6.

Section 9: new, except as detailed in the Historical Note 9.18 and in the paragraph numbered 9.27. I would draw attention to Example 9.32. The exhaustive character of the set of results given in the Peroration that begins on page 224 is again a consequence of a systematic minimisation, during the final revision, of assumptions in proofs throughout the paper.

Section 10: the views expressed in the essay are mine.

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